# A conjecture by Galatius and Randal-Williams:
## A synthetic approach after Burklund, Hahn and Senger.

João Fernandes and Jan McGarry

**Abstract**

In this report, we study the recent work of Burklund, Hahn, and Senger in [BHS19], on a classification problem of highly connected manifolds. The goal of this project was to understand the classical study of the problem, and also the referenced novel approach to it. This study was done as a *Project outside courses' scope* (PUK) during Block 2 of the course 2021-2022 at the University of Copenhagen, under the supervision of Søren Galatius.

## Contents



## 1. Introduction

The problem of the classification of closed, $(n-1)$-connected, $2n$ dimensional smooth manifolds has been one of the success stories of algebraic and geometric topology in the last 60 years. This problem was first introduced and studied by Wall in [Wal67]. His key insight was to instead remove a disc from our objects of interest and classify compact, $(n-1)$-connected, $2n$ smooth manifolds with boundary homeomorphic to a sphere. These objects were then fully classified in [Wal67] by algebraic data (called an *n-space*). The final piece of the puzzle was to classify all the different ways one could glue the removed disc back.

Meanwhile, in [KM63] Kervaire and Milnor found exotic spheres, that is smooth manifolds which are homeomorphic but not diffeomorphic to the standard sphere $m$-sphere. Moreover, they proved that the set of such objects $\Theta_m$ forms a group under connected sum, and found an exact sequence, which in our case $m = 2n-1$ is of the form

$$0 \longrightarrow \mathrm{bP}_{2n} \longrightarrow \Theta_{2n-1} \longrightarrow \pi_{2n-1}\mathbb{S}/\operatorname{im} J.$$

This provided a new formulation for the missing step in Wall's program. We can rephrase his question to finding which exotic spheres can be boundaries of compact, $(n-1)$-connected, $2n$-manifolds.

## 1.1. **The work of Stolz and the reduction to homotopy theory**

This problem was greatly developed in the work of Stolz in [Sto85]. His key insight was to use (relative) Pontryagin-Thom theory to reduce the problem to a purely homotopy theoretic one. Let us describe his approach.

Let $W$ be a compact, $(n-1)$-connected, $2n$-manifold such that $\partial W = \Sigma \in \Theta_{2n-1}$. In [KM63], Kervaire and Milnor proved that every exotic sphere is stably paralelizable. In other words, the classifying map for the stable tangent bundle of a homotopy sphere $\tau_\Sigma \colon \Sigma \to \mathrm{BO}$ admits a lift $l_\Sigma \colon \Sigma \to \mathrm{EO}$. Consequently, such a pair $(W, \Sigma)$ will induce the following commutative diagram of spaces

$$\begin{tikzcd} \Sigma \arrow[r, "l_\Sigma"] \arrow[d, hook] & \mathrm{EO} \arrow[d] \\ W \arrow[r, "\tau_W"] & \mathrm{BO}. \end{tikzcd}$$

We can see that the inclusion $\Sigma \hookrightarrow W$ is an $(n-1)$-connected map. By obstruction theory, since the pair $(\Sigma, W)$ is $(n-1)$-connected and the truncation map $\tau_{\geq n}\mathrm{BO} \to \mathrm{BO}$ is $n$-coconnected it follows that all obstruction classes for a lift $W \to \tau_{\geq n}\mathrm{BO}$ vanish. Thus we have the following partial lift.

$$\begin{tikzcd} \Sigma \arrow[r, "l_\Sigma"] \arrow[dd, hook] & \mathrm{EO} \arrow[d] \\ & \tau_{\geq n}\mathrm{BO} \arrow[d] \\ W \arrow[r, "\tau_W"] \arrow[ur, dashed, "l_W"] & \mathrm{BO}. \end{tikzcd}$$

The pair $(W, \Sigma)$ equipped with the lifts $l_W$ and $l_\Sigma$, represents an element of the relative cobordism group $\Omega_{2n}^{\tau_{\geq n}\mathrm{BO},\mathrm{EO}}$, for which we use the notation $\Omega_{2n}^{\langle n\rangle,\mathrm{fr}}$. The relative Pontrjagin-Thom construction gives an equivalence

$$\Omega_{2n}^{\langle n\rangle,\mathrm{fr}} \xrightarrow{\simeq} \pi_{2n}(\mathrm{MO}\langle n\rangle, \mathbb{S}).$$

Under this correspondence, the forgetful map taking $[W, \Sigma, l_W, l_\Sigma]$ to $[\Sigma, l_\Sigma] \in \Omega_{2n-1}^{\mathrm{fr}}$ corresponds to the boundary map

$$\pi_{2n}(\mathrm{MO}\langle n\rangle, \mathbb{S}) \to \pi_{2n-1}(\mathbb{S}).$$

Therefore, our problem is reduced to finding the image of this map, which is equal to the kernel of the unit map

$$\pi_{2n-1}\mathbb{S} \to \pi_{2n-1}\mathrm{MO}\langle n\rangle$$

in the long exact sequence. The strategy of Stolz was to analyse the kernel of this map via the Adams spectral sequence.

### 1.2. **Recent developments by Galatius – Randall-Williams, and Burklund – Hahn – Senger**

The approach of Stolz in [Sto85] was very succesful, but left some cases unknown. For example, it was concluded that if $n$ is congruent to 1 modulo 4, then this kernel was at most generated by one element modulo the image of $J$. Many decades later, Galatius and Randall-Williams conjectured in their work calculating abelianisations of mapping class groups [GR16, Conjecture A] that this element was in fact trivial modulo the image of $J$. A proof of this conjecture in many cases was obtained in recent work of Burklund, Hahn and Senger [BHS19], and it is the centre of this report.

**Theorem 1.1** ([BHS19])**.** *For all natural numbers $n > 31$, the kernel of the unit map*

$$\pi_{8n-1}\mathbb{S} \to \pi_{8n-1}\,\mathrm{MO}\langle 4n\rangle$$

*is the image of the $J$-homomorphism.*

Their key insight was to use recent developments in stable homotopy theory to encode and keep track of much of the same data on the Adams spectral sequence that Stolz was working with. This was possible due to work by Pstrągowsky in [Pst18] introducing synthetic spectra. The goal of this report is to explore different aspects of this new approach to prove results such as the above.

To conclude, we return to our main motivating problem. By the Milnor-Kervaire exact sequence, we know that if $\Sigma \in \Theta_{8n-1}$ lies in the kernel of the last map, then it came from $\mathrm{bP}_{8n}$ Therefore, we have the following corollary.

**Corollary 1.2.** *Let $n > 31$, and $W$ be a compact, $(4n-1)$-connected, $8n$ dimensional smooth manifold with boundary. If $\partial W$ is homemorphic to a sphere, then it bounds a parallelizable manifold.*

### 1.3. **Outline of the report**

In Section 2, we explore a modern definition of Thom spectra based on the work of Ando, Blumberg, Gepner, Hopkins and Rezk in [And+14]. This definition and the consequent description as a bar construction in 4 will be the starting point of the approach to the study of the unit map that we follow. We see two advantages of this description. Firstly, from this bar construction we obtain a spectral sequence converging to the homotopy of its realisation, namely the Thom spectrum. Secondly, the associated graded terms of the filtration by skeleta are increasily more connected, and, therefore, for our range of interest we can restrict to the second step of this filtration.

In Section 3.1, we study this spectral sequence in the desired range, and by referencing crucial calculations done in [BHS19], we will be able to reduce the study to a single differential in the second page. In order to do this, we will describe this differential as a Toda bracket; we provide a systematic way of doing so in Section 3.2. The goal is then to find a lower bound for the Adams filtration of this bracket that implies that it lies in the image of the $J$-homomorphism. In the work of Burklund, Hahn and Senger in [BHS19], the use of Pstrągowsky's syntethic spectra facilitated the study of Adams filtrations of homotopies and Toda brackets. In Section 3.4, we present the theory of synthetic spectra needed from [Pst18] and [BHS19, Chapter 9]. We finish the proof in Section 3.6 by using synthetic spectra along with classical and modern upper bounds on non-trivial elements in $\mathrm{coker}(J)$.

## 2. Thom spaces and Thom spectra

**Definition 2.1.** Let $\xi\colon E \to B$ be a spherical fibration together with a section $s$. Define the Thom space $\mathrm{Th}(\xi)$ to be the quotient space $E/B = \mathrm{cof}(s)$.

**Example 2.2.** Let $\xi\colon E \to X$ be a rank $n$ vector bundle. Apply one-point compactification to each fiber, and denote the obtained spherical fibration by $S^\xi \to B$. The points at infinity specify a section $s\colon B \to S^\xi$. The space $\mathrm{Th}(S^\xi)$ is called the Thom space of $\xi$ and denoted by $\mathrm{Th}(\xi)$.

Now we show how we can build a spectrum from a vector bundle, or more generally a virtual vector bundle. Notice that Thom spaces are canonically pointed. Let $f : B \to BO$ be a map of spaces, and for each natural number $n$ define $B_n$ as the pullback

$$\begin{tikzcd} B_n \arrow[r] \arrow[d, "f_n"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "f"] \\ BO(n) \arrow[r] & BO. \end{tikzcd}$$

Let $f_n^*\gamma^n$ be the vector bundle classified by $f_n$, where $\gamma^n$ denotes the universal rank $n$-vector bundle. Notice that the following pullback square at the left

$$\begin{tikzcd} B_n \arrow[r] \arrow[d, "f_n"] \arrow[dr, phantom, "\lrcorner", very near start] & B_{n+1} \arrow[r] \arrow[d, "f_{n+1}"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "f"] \\ BO(n) \arrow[r] & BO(n+1) \arrow[r] & BO. \end{tikzcd}$$

induces a map of vector bundles $f_n^*\gamma^n \oplus \epsilon^1 \to f_{n+1}^*\gamma_{n+1}$, for the stabilization map $BO(n) \to BO(n+1)$ induces a map $\gamma^n \oplus \epsilon^1 \to \gamma^{n+1}$, and this is pulled back along $f_n$ and $f_{n+1}$.

For any vector bundle $\xi$, there is a homeomorphism $\Sigma\,\mathrm{Th}(\xi) \cong \mathrm{Th}(\xi \oplus \epsilon^1)$. Therefore, the sequence of Thom spaces $\{\mathrm{Th}(f_n^*(\gamma^n)\}_{n\in\mathbb{N}}$ along with the stabilization maps above form a prespectrum.

**Definition 2.3.** We call the prespectrum $\{\mathrm{Th}(f_n^*\gamma^n)\}_{n\in\mathbb{N}}$ the *Thom prespectrum* of $f : B \to BO$. We call the associated spectrum

$$Mf := \mathrm{colim}_n\, \Sigma^{\infty-n}(\mathrm{Th}(f_n^*\gamma^n))$$

the *Thom spectrum* of $f$.

For example, if a vector bundle $\xi\colon E \to B$ is classified by a map $f : B \to BO(n)$, then the Thom spectrum of $f : B \to BO(n) \to BO$ is the suspension spectrum of $\mathrm{Th}(\xi)$.

**Definition 2.4.** We call the Thom spectrum of the identity $\mathrm{id}\colon BO \to BO$ the *universal Thom spectrum*, denoted MO.

In Section 3, we will use another description of Thom spectra. For this, we first introduce the $J$-homomorphism.

### 2.1. The $J$-homomorphism

For every natural numbers $n$ and $i \geq 2$, the classical $J$-homomorphism is a homomorphism $\pi_i SO(n) \to \pi_i O(n) \to \pi_{n+i} S^n$, constructed via the Hopf construction. Following [Ati66], we can describe this map in the following way. For a natural number $n$, write $H_n$ for the group of homotopy self-equivalences of $S^n$ which preserve the point at infinity. Since every orthogonal transformation $\mathbb{R}^n \to \mathbb{R}^n$ extends to a homeomorphism of $(\mathbb{R}^n)^+ \cong S^n$ onto itself, we have a natural map

$$O(n) \longrightarrow H_n \hookrightarrow \Omega^n S^n;$$

The second map is the inclusion of the two connected components of $\Omega^n S^n$ containing id and $-$id, respectively. Taking homotopy groups we get a morphism

$$\pi_i O(n) \longrightarrow \pi_i \Omega^n S^n \cong \pi_{n+i} S^n.$$

This is the (unstable) *$J$-homomorphism*. However, notice that the maps $O(n) \to H_n$ are compatible with the stabilization maps $O(n) \to O(n+1)$, and the suspension morphism $\pi_{n+i}S^n \to \pi_{n+1+i}S^{n+1}$. Hence, taking colimits we get a map

$$J : O \longrightarrow H \simeq \mathrm{GL}_1(\mathbb{S})$$

which on homotopy groups recovers the *stable $J$-homomorphism*

$$J : \pi_i O \longrightarrow \pi_i \mathbb{S}.$$

Here, $\mathrm{GL}_1(\mathbb{S})$ is the $\infty$-group of units of $\mathbb{S}$, as in [And+14]. A model for this $\mathbb{E}_\infty$ space is as the connected components of $\Omega^\infty \mathbb{S}$ containing id and $-$id, equipped with its multiplicative structure coming from $\mathbb{S}$. Note that $\pi_i \mathrm{GL}_1(\mathbb{S}) \simeq \pi_i \mathbb{S}$.

We can even realize $J$ as a map of infinite loop spaces, and thus $J$ deloops once to

$$BJ : BO \longrightarrow BH \simeq B\,\mathrm{GL}_1(\mathbb{S}).$$

Since $B\,\mathrm{GL}_1(\mathbb{S})$ classifies stable spherical fibrations, this map is the universal in the sense that the classifying map of any (stable) spherical fibration which comes from a vector bundle factors through $J$.

## 2.2. **Thom spectra as a tensor product**

We now write the Thom space of a vector bundle in a different way, which we will use for Section 3. Let $\xi : E \to B$ be a rank $n$ vector bundle, and $f : B \to BO(n)$ its classifying map. Then $\xi$ is equivalent to the vector bundle $\xi' \times_{O(n)} \mathbb{R}^n$, associated to the principal $O(n)$-bundle defined by the pullback

$$\begin{tikzcd} \xi' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & EO(n) \arrow[d] \\ X \arrow[r, "f"] & BO(n). \end{tikzcd} \tag{1}$$

Then the spherical fibration associated to $\xi$ may be rewritten as $\xi' \times_{O(n)} S^n$, where the second factor is the one-point compactification of $\mathbb{R}^n$ [1]. Now, quotienting out with the section at infinity is the same thing as adding a disjoint basepoint to $\xi'$ and then forming the smash product $\xi'_+ \wedge_{O(n)} S^n$. Indeed, the later amounts to quotienting the subspace $\xi'_+ \times \{\infty\}$ of $\xi'_+ \times_{O(n)} S^n$, which is compatible with the base-point preserving acion of $O(n)$ on $S^n$; and then also quotienting the copy $\{+\} \times S^n$ we have added to the original product. We have shown that

$$\mathrm{Th}(\xi) \simeq \xi'_+ \wedge_{O(n)} S^n. \tag{2}$$

Let us look at the modern definition due to Ando, Blumberg, Gepner, Hopkins and Rezk in [And+14].

**Definition 2.5.** Let $f : B \to BO$ be a map of infinite loop spaces. The *Thom spectrum $Mf$* of $f$ is defined as the homotopy pushout of the following diagram in the model category Elmendorf–Mandell–Kriz–May commutative $\mathbb{S}$-algebras [Elm+97]:

$$\begin{tikzcd} \Sigma^\infty_+ \mathrm{GL}_1 \mathbb{S} \arrow[r] \arrow[d] & \mathbb{S} \arrow[d] \\ \Sigma^\infty_+ \mathrm{hofib}(f) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & M. \end{tikzcd}$$

[1]We have extended the action of $O(n)$ using that an orthogonal transformation $\mathbb{R}^n \to \mathbb{R}^n$ extends to a homeomorphism of $(\mathbb{R}^n)^+ \cong S^n$ onto itself. Note that this action preserves the base-point at infinity

*Remark* 2.6. Given a map $f : B \to BO$, we know that

$$\begin{tikzcd} \xi' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & EO \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E\,\mathrm{GL}_1(\mathbb{S}) \arrow[d, "\mathrm{can}"] \\ B \arrow[r, "f"] & BO \arrow[r, "J"] & B\,\mathrm{GL}_1(\mathbb{S}). \end{tikzcd} \tag{3}$$

Therefore in the above definition, $\mathrm{hofib}(f) \simeq \xi'$, as in 1. We obtain another description of $\mathrm{M}f$:

$$\begin{tikzcd} \Sigma^\infty_+ B \arrow[r] \arrow[d] & \Sigma^\infty_+ \mathrm{GL}_1\,\mathbb{S} \arrow[r] \arrow[d] & \mathbb{S} \arrow[d] \\ \Sigma^\infty_+ * \simeq \mathbb{S} \arrow[r] & \Sigma^\infty_+ \mathrm{hofib}(f) \arrow[r] \arrow[ul, phantom, "\ulcorner", very near start] & \mathrm{M}f \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

By [Elm+97, Proposition 1.6], this homotopy pushout expresses $Mf$ as the derived tensor product

$$Mf \simeq \Sigma^\infty_+ \mathrm{hofib}(f) \wedge_{\Sigma^\infty_+ \mathrm{GL}_1 \mathbb{S}} \mathbb{S};$$

cf. equation 2. This is the model we will use throughout the rest of this report.

## 3. On the boundaries of highly connected manifolds

The goal of this section is to prove:

**Theorem 3.1** ([BHS19])**.** *For all natural numbers $n > 31$, the kernel of the unit map*

$$\pi_{8n-1}\mathbb{S} \to \pi_{8n-1}\,\mathrm{MO}\langle 4n\rangle$$

*is the image of the $J$-homomorphism.*

First, we will reduce the proof of the Theorem to the vanishing of a differential of a Bar spectral sequence for $\mathrm{MO}\langle 4n\rangle$ – Lemma 3.5 –; then we will reduce this to the vanishing of a Toda bracket $\omega \in \pi_{8n-1}\mathbb{S}/\operatorname{im} J$ – Lemma 3.8 –; and, finally, we will study this Toda bracket using synthetic spectra to conclude that it vanishes with the dimension assumptions in the statement above.

Following [BHS19], we will study the map $\pi_{8n-1}\mathbb{S} \to \pi_{8n-1}\,\mathrm{MO}\langle 4n\rangle$ via the spectral sequence associated to the bar-construction realizing the model for the Thom spectrum $\mathrm{MO}\langle 4n\rangle$ given in [And+14, Definition 4.1], as the following homotopy pushout in the model category Elmendorf–Mandell–Kriz–May commutative $\mathbb{S}$-algebras [Elm+97]

$$\begin{tikzcd} \Sigma^\infty_+\Omega^\infty\Omega\tau_{\geq 4n}ko \arrow[r] \arrow[d] & \Sigma^\infty_+\Omega^\infty\,\mathrm{gl}_1(\mathbb{S}) \arrow[r] & \mathbb{S} \arrow[d] \\ \Sigma^\infty_+\Omega^\infty * \simeq \mathbb{S} \arrow[rr] & & \mathrm{MO}\langle 4n\rangle. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The top-left horizontal map is the composition of the desuspension of a $(4n-1)$-truncation of $ko$ with the $J$-homomorphism – precisely the infinite delooping of $BJ$ described in Section 2.1 –

$$\Omega^\infty\Omega\tau_{\geq 4n}ko \longrightarrow \Omega^\infty\Omega ko \xrightarrow{J} \Omega^\infty\,\mathrm{gl}_1(\mathbb{S}).$$

If we denote by $J_+$ and $\epsilon$ the maps of $\mathbb{E}_\infty$ rings in the pushout diagram, then the above takes the form

$$\begin{tikzcd} \Sigma^\infty_+ O\langle 4n-1\rangle \arrow[r, "J_+"] \arrow[d, "\epsilon"'] & \mathbb{S} \arrow[d] \\ \mathbb{S} \arrow[r] & \mathrm{MO}\langle 4n\rangle, \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

where $O\langle 4n-1\rangle := \Omega^\infty \Omega \tau_{\geq 4n} ko$. This notation should cause no confusion for one can observe that this space is equivalent to the $(4n-1)$-truncation of $O$.

By [Elm+97, Proposition 1.6], the homotopy pushout is the derived tensor product, which can be modeled by the bar-construction: $\mathrm{MO}\langle 4n\rangle \simeq |\mathrm{Bar}(\mathbb{S}, \Sigma^\infty_+ O\langle 4n-1\rangle, \mathbb{S})|$, where the face maps of Bar take the form

$$\cdots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \Sigma^\infty_+ O\langle 4n-1\rangle^{\otimes 2} \underset{\epsilon\otimes 1}{\overset{1\otimes J_+}{\underset{m}{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}}}} \Sigma^\infty_+ O\langle 4n-1\rangle \underset{\epsilon}{\overset{J_+}{\substack{\longrightarrow\\ \longrightarrow}}} \mathbb{S}, \tag{4}$$

where $m$ denotes the multiplication map given by the $\mathbb{E}_\infty$ structure of $\Sigma^\infty_+ O\langle 4n-1\rangle$.

### 3.1. The Bar spectral sequence

Consider the skeletal filtration of the simplicial spectrum given by the bar construction,

$$\mathbb{S} \longrightarrow \mathrm{Bar}_{\leq 1} \longrightarrow \mathrm{Bar}_{\leq 2} \longrightarrow \cdots \to \mathrm{MO}\langle 4n\rangle.$$

In this section we will study the map $\pi_{8n-1}\mathbb{S} \to \pi_{8n-1}\mathrm{MO}\langle 4n\rangle$ via a spectral sequence induced by this filtration, and reduce the main theorem 3 to the vanishing of a differential of the spectral sequence 3.5.

For a general spectrum $X$ with a filtration $X = \cup_s F_s X$, we consider the spectral sequence with signature

$$\mathrm{E}^1_{s,t} = \pi_{s+t}(F_s X / F_{s-1} X) \Longrightarrow \pi_s X.$$

For a skeletal filtration arising from a simplicial spectrum $X_\bullet$, we can identify the cofibres so that the entries in the first page are

$$\mathrm{E}^1_{s,t} = \pi_t(X_s) \Longrightarrow \pi_s |X|,$$

and the $t$-th row $E^1_{\bullet,t}$ identifies with the normalised complex of the simplicial abelian group $\pi_t X_\bullet$.

We will now show that the associated graded terms of the filtration 3.1 are highly connected. Since we are interested in the map $\pi_{8n-1}\mathbb{S} \to \pi_{8n-1}\mathrm{MO}\langle 4n\rangle$ in homotopy groups of low degrees, this will allow us to restrict our study of it to $\mathrm{Bar}_{\leq 2}$.

**Lemma 3.2.** *The canonical map* $\mathrm{Bar}_{\leq 2} \to \mathrm{MO}\langle 4n\rangle$ *is* $(12n-1)$*-connected.*

*Proof.* The result follows because the maps $\mathrm{Bar}_{\leq k-1} \to \mathrm{Bar}_{\leq k}$ are $(4nk-1)$-connected. To show this, we fix $k \geq 1$, and show that the cofibre $\Sigma^k \Sigma^\infty_+ O\langle 4n-1\rangle^{\otimes k}$ of this map is $(4nk-1)$-connected.

First of all, $\Sigma^\infty_+ O\langle 4n-1\rangle$ is $(4n-2)$-connected because $O\langle 4n-1\rangle$ is. Its $k$-th tensor power is $(4nk-k-1)$-connected because $\Sigma^\infty_+ : \mathrm{Spaces} \to \mathcal{S}\mathrm{p}$ is symmetric monoidal, and if $X, Y$ are pointed spaces which are $n$-connected and $m$-connected respectively, then $X \wedge Y$ is $(n+m+1)$-connected. Finally, taking suspension $k$ times adds $k$-degrees of connectivity. Thus, we conclude that $\Sigma^k \Sigma_+ O\langle 4n-1\rangle^{\otimes k}$ is $(4nk-1)$-connected. □

Consequently, for our interests, we may only consider the spectral sequence induced by the filtration $\mathbb{S} \to \mathrm{Bar}_{\leq 1} \to \mathrm{Bar}_{\leq 2}$, and it takes the following form in the $\mathrm{E}^1$-page, concentrated in three rows:

$$
\begin{array}{c|ccc}
8n-1 & \pi_{8n-1}\mathbb{S} \xleftarrow{J} & \pi_{8n-1}\Sigma^\infty O\langle 4n-1\rangle & \cdots \\
8n-2 & \pi_{8n-2}\mathbb{S} \xleftarrow{J} & \pi_{8n-2}\Sigma^\infty O\langle 4n-1\rangle \xleftarrow{1\otimes J-m} & \pi_{8n-2}\Sigma^\infty O\langle 4n-1\rangle^{\otimes 2} \\
8n-3 & \cdots & \cdots & \pi_{8n-3}\Sigma^\infty O\langle 4n-1\rangle^{\otimes 2} \\
\hline
 & 0 & 1 & 2
\end{array}
$$

In [BHS19, §4], the authors calculate these groups, using Goodwillie towers. We summarize these calculations in the following result.

**Proposition 3.3.** *Let $x$ be a fixed generator of $\pi_{4n-1}\Sigma^\infty O\langle 4n-1\rangle \simeq \mathbb{Z}$. The following statements hold:*

1. *The spectrum $\Sigma^\infty O\langle 4n-1\rangle^{\otimes 2}$ is $(8n-3)$-connected.*
2. *The group $\pi_{8n-2}\Sigma^\infty O\langle 4n-1\rangle^{\otimes 2}$ is isomorphic to $\mathbb{Z}\{x\otimes x\}$ where $x \in \pi_{4n-1}\Sigma^\infty O\langle 4n-1\rangle \simeq \mathbb{Z}$ is a generator.*
3. *The group $\pi_{8n-2}\Sigma^\infty O\langle 4n-1\rangle$ is cyclic of order 2 and generated by $x^2$,*
4. *The element $xJ(x)$ is 0 in $\pi_{8n-2}\Sigma^\infty O\langle 4n-1\rangle$.*

*Proof.* The first claim follows from the fact that $A\otimes B$ is $(n+m+1)$-connected if $A$ and $B$ are $n$ and $m$-connected spectra, respectively. The second claim stems from the Hurewicz isomorphism theorem; the third is [BHS19, Corollary 4.8]; and the forth is [BHS19, Lemma 4.10]. □

Therefore, we can simplify the above spectral sequence as shown below.

$$
\begin{array}{c|ccc}
8n-1 & \pi_{8n-1}\mathbb{S} \xleftarrow{J} & \pi_{8n-1}\Sigma^\infty O\langle 4n-1\rangle & \cdots \\
8n-2 & \pi_{8n-2}\mathbb{S} \xleftarrow{J} & \mathbb{Z}/2\{x^2\} \xleftarrow{1\otimes J-m} & \mathbb{Z}\{x\otimes x\} \\
8n-3 & \cdots & \cdots & 0 \\
\hline
 & 0 & 1 & 2
\end{array}
$$

By the forth result in 3.3, we see that the kernel of $1\otimes J - m$ is generated by $2(x\otimes x)$ and the map is surjective. To conclude the computation of the $E^2$ page in this range, we present the following result that resolves the possible confusion between the map $J$ and the classical $J$-homomorphism.

**Theorem 3.4** ([BHS19, Theorem 4.11])**.** *If $4n-1\le k\le 8n-1$, then the image of*

$$J\colon \pi_k\Sigma^\infty O\langle 4n-1\rangle \longrightarrow \pi_k\mathbb{S}$$

*agrees with the image of the classical $J$-homomorphism described in Section 2.1.*

Now we can identify how the entries of interest look in the $E^2$ page of the spectral sequence, as in the following diagram.

$$\begin{tikzcd}
8n-1 & \mathrm{coker}(J)_{8n-1} & \cdots & \cdots \\
8n-2 & \mathrm{coker}(J)_{8n-2} & 0 & \mathbb{Z}\{2(x\otimes x)\} \arrow[ull, "d^2"'] \\
8n-3 & \cdots & \cdots & 0 \\
 & 0 & 1 & 2
\end{tikzcd}$$

We conclude this reduction with the following Lemma.

**Lemma 3.5.** *Let $n$ be a natural number. The kernel of the unit map $\pi_{8n-1}\mathbb{S} \to \pi_{8n-1}\,\mathrm{MO}\langle 4n\rangle$ is the image of the $J$-homomorphism if the differential $d^2\colon \mathbb{Z}\{2(x\otimes x)\} \to \mathrm{coker}(J)_{8n-1}$ in the Bar spectral sequence for $\mathrm{MO}\langle 4n\rangle$ vanishes.*

*Proof.* Note that no other non-trivial higher differentials apart from the depicted above will hit the line of dimension $8n-1$, on which only $\mathrm{coker}(J)_{8n-1}$ is left. Therefore, by convergence of the spectral sequence, the edge homomorphism

$$\mathrm{coker}(J)_{8n-1}/d^2 \simeq E^3_{0,8n-1} \longrightarrow E^\infty_{0,8n-1} \simeq \pi_{8n-1}\,\mathrm{Bar}_{\leq 2} \simeq \pi_{8n-1}\,\mathrm{MO}\langle 4n\rangle$$

is an isomorphism.

By construction of the spectral sequence, the above map is induced on quotients by the map

$$F_0(\pi_{8n-1}\,\mathrm{MO}\langle 4n\rangle) \longrightarrow E^\infty_{0,8n-1},$$

from the 0-th filtration step of the target to the $E^\infty$ page; that is, $F_0(\pi_{8n-1}\,\mathrm{MO}\langle 4n\rangle)$ is the image of the map induced by the inclusion of the 0-th skeleton $\mathbb{S}$, which coincides with the unit map because they are both maps in the category of $\mathbb{S}$-algebras, where $\mathbb{S}$ is initial.

All in all, the kernel of the unit map in the statement is the kernel of the quotient map

$$\pi_{8n-1}\mathbb{S} \simeq E^1_{0,8n-1} \longrightarrow E^3_{0,8n-1} \simeq \mathrm{coker}(J)_{8n-1}/d^2,$$

which is exactly $J$ if and only if $d^2$ vanishes. □

### 3.2. **Toda brackets as spectral sequence differentials**

The goal of this section is to provide a model for the element $d_2(2(x\otimes x)) \in \mathrm{coker}(J)_{8n-1}$. However, we may work in higher generality. Let $X$ be a spectrum and

$$\cdots \longrightarrow F_{s-1}X \longrightarrow F_sX \longrightarrow F_{s+1}X \longrightarrow \cdots \longrightarrow X$$

a filtration on $X$. The associated spectral sequence takes the form

$$\mathrm{E}^1_{s,t} = \pi_{s+t}(F_sX/F_{s-1}X) \Longrightarrow \pi_sX.$$

We can describe $d^1 : \mathrm{E}^1_{s,t} \to \mathrm{E}_{s-1,t}$ as induced under $\pi_{s+t}$ by the map of spectra

$$d^1\colon F_s/F_{s-1} \xrightarrow{k} \Sigma F_{s-1} \xrightarrow{j} \Sigma(F_{s-1}/F_{s-2}),$$

where $k$ is the connecting homomorphism, and $j$ is the quotient map. To describe the second differential $d^2 : \mathrm{E}^2_{s,t} \to \mathrm{E}^2_{s-2,t+1}$, let $x \in E^1_{s,t}$ such that $d^1 \circ x \simeq 0$, that is, it has a lift to $\Sigma F_{s-2}$ as depicted below.

$$\begin{tikzcd}
\mathbb{S}^{s+t} \arrow[r, dashed, "\tilde{x}"] \arrow[dr, "x"'] & \operatorname{hofib} d^1 \arrow[r, dashed, "can"] \arrow[d] & \Sigma F_{s-2} \arrow[r, "j"] \arrow[d] & \Sigma(F_{s-2}/F_{s-3}) \\
& F_s/F_{s-1} \arrow[r, "k"] \arrow[dr, "d^1"'] & \Sigma F_{s-1} \arrow[d, "j"] & \\
& & \Sigma(F_{s-1}/F_{s-2}), &
\end{tikzcd} \tag{5}$$

where $can$ is the canonical map to the homotopy fiber of $j$. Then $d^2([x]) \in E^2_{s-2,t+1}$ can be represented by the composite in the top row in the picture above. We next show how we can model $d^2([x])$ as a Toda bracket.

**Construction 3.6.** Consider the sequence of maps

$$\mathbb{S}^{s+t} \xrightarrow{x} F_sX/F_{s-1}X \xrightarrow{d^1} \Sigma(F_{s-1}X/F_{s-2}X) \xrightarrow{d^1} \Sigma^2(F_{s-2}X/F_{s-3}X). \tag{6}$$

By assumption, the composition $d^1 \circ x$ is nullhomotopic; pick a nullhomotopy $f : d^1 \circ x \simeq 0$. We also know that $d^1 \circ d^1$ is nullhomotopic; moreover, there is a canonical nullhomotopy of $d^1 \circ d^1 = j \circ k \circ j \circ k$, namely $a : k \circ j \simeq 0$. We will associate to this data a map $z : \mathbb{S}^{s+t} \to \Sigma(F_{s-2}/F_{s-3})$, from the source to loops of the target.

With the data above, we can lift the maps in 6 as follows.

$$\begin{tikzcd}
& \operatorname{hofib}(d_1) \arrow[r, "can"] \arrow[d] & \Sigma F_{s-2}X \arrow[d, "i"'] & \operatorname{hofib}(d_1) \arrow[d] & \\
\mathbb{S}^{s+t} \arrow[ur, bend left] \arrow[r, "x"] & F_sX/F_{s-1}X \arrow[r, "k"] \arrow[u, phantom, "\Leftarrow \tilde{f}"] & \Sigma F_{s-1}X \arrow[r, "j"] \arrow[ur, bend left] & \Sigma(F_{s-1}X/F_{s-2}X) \arrow[r, "j\circ k = d^1"] \arrow[u, phantom, "\Leftarrow \tilde{a}"] & \Sigma^2(F_{s-2}X/F_{s-3}X).
\end{tikzcd}$$

Moreover, there is also a nullhomotopy of the map

$$\Sigma F_{s-2} \xrightarrow{i} \Sigma F_{s-1} \longrightarrow \operatorname{hofib} d^1 \longrightarrow \Sigma(F_{s-2}/F_{s-3}),$$

namely a composition of $\tilde{a}$ and the canonical for $j \circ i \simeq 0$. Therefore, we can similarly produce a lift $j'$ to the homotopy fiber of $\operatorname{hofib}(d^1) \to \Sigma(F_{s-1}/F_{s-2})$, that is $\Sigma(F_{s-2}/F_{s-3})$, and obtain the following folded diagram

$$\begin{tikzcd}
& & \Sigma(F_{s-2}X/F_{s-3}X) \arrow[d] & \\
& \operatorname{hofib} d_1 \arrow[ur, "j'\circ can"] \arrow[d] & \operatorname{hofib} d_1 \arrow[d] & \\
\mathbb{S}^{s+t} \arrow[ur] \arrow[r, "x"] & F_sX/F_{s-1}X \arrow[ur] \arrow[r, "d^1"] & \Sigma(F_{s-1}X/F_{s-2}X) \arrow[r, "d^1"] & \Sigma^2(F_{s-2}X/F_{s-3}X).
\end{tikzcd} \tag{7}$$

We consider the map

$$z : \mathbb{S}^{s+t} \longrightarrow \operatorname{hofib} d_1 \xrightarrow{j'\circ can} \Sigma(F_{s-2}X/F_{s-3}X),$$

and refer to it as a *Toda bracket*. Now we show that $z$ represents $d^2([x]) \in E^2_{s-2,t+1}$.

**Lemma 3.7.** *In the above construction, the element $[z]$ is equal to $d_2([x])$ in $E^2_{s-2,t+1}$, and is independent of the homotopy $d^1 \circ x \simeq 0$ chosen.*

*Proof.* By examining the description of $d^2$ in 5 and the result 7 of the above construction, to prove the first part of the statement it suffices to show that the cofibre map $\Sigma F_{s-2}X \to \Sigma(F_{s-2}X/F_{s-3}X)$ in 5 is homotopic to the lift $j'$ in 7. Notice that by construction, $j'$ is obtained as the map induced on fibers as follows:

$$\begin{tikzcd}
\Omega A \arrow[r] \arrow[d, "j'"] & \mathrm{hofib}(k) \arrow[r] \arrow[d] & Y \arrow[r, "k"] \arrow[d, equal] & A \arrow[d, "j"] \\
\Omega Z \arrow[r] & \mathrm{hofib}(d^1) \arrow[r] & Z \arrow[r, "d^1"] & Z,
\end{tikzcd}$$

where we write $Y \to A \to Z$ for

$$d^1 \colon \Sigma(F_{s-1}X/F_{s-2}X) \xrightarrow{k} \Sigma^2 F_{s-2}X \xrightarrow{j} \Sigma^2(F_{s-2}X/F_{s-3}X).$$

In other words, it is the map induced by $j$ between the corresponding stages in the Puppe sequences, that is, $\Omega j$. This concludes the proof of the first part.

Finally, for the second part of the statement, notice that another choice of homotopy $d^1 \circ x \simeq 0$ only differs by a loop in $\Sigma(F_{s-1}/F_{s-2})$, so that the lift differs by a map $\mathbb{S}^{s+t} \to F_{s-1}/F_{s-2}$, which composed with $d^1$ represents an element of $E^2_{s-2,t+1}$. However, classes in the image of $d^1$ have been quotiented out in $E^2$ by construction. $\square$

## 3.3. The problem as a Toda bracket

We apply Lemma 3.7 to the Bar spectral sequence for $\mathrm{MO}\langle 4n\rangle$. In particular, we can model the element $d^2(2(x \otimes x)) \in \mathrm{coker}(J)_{8n-1}$ as the Toda bracket $[z]$ associated to the diagram

$$\begin{tikzcd}
\mathbb{S}^{8n-2} \arrow[r, "2(x\otimes x)"] \arrow[rr, bend left, ""{name=U}] & \Sigma^\infty O\langle 4n-1\rangle^{\otimes 2} \arrow[r, "1\otimes J - m"] \arrow[rr, bend right, ""{name=D}] \arrow[phantom, from=U, to=2-2, "f\Updownarrow"] & \Sigma^\infty O\langle 4n-1\rangle \arrow[r, "J"] \arrow[phantom, d, "\Updownarrow a"] & \mathbb{S},
\end{tikzcd} \tag{8}$$

where $f$ is an arbitrary nullhomotopy $d^1 \circ 2(x \otimes x) \simeq 0$, and $a$ is the canonical nullhomotopy $d^1 \circ d^1 \simeq 0$. Similarly to Section 3.1, we are identifying the differentials of the $E^1$ page of the spectral sequence at the level of spectra with those of the normalized filtered spectrum. We claim that, under this identification, the canonical nullhomotopy $d^1 \circ d^1 \simeq 0$ corresponds to the canonical nullhomotopy $J \circ (1 \otimes J - m) \simeq 0$ given by the structure on $J$ as a map of (non-unital) $\mathbb{E}_\infty$ rings[2]. Therefore:

**Lemma 3.8.** *Let $n$ be a natural number. The kernel of the unit map $\pi_{8n-1}\mathbb{S} \to \pi_{8n-1}\mathrm{MO}\langle 4n\rangle$ is the image of the $J$-homomorphism if the Toda bracket $[z] \in \mathrm{coker}(J)_{8n-1}$ associated to diagram 8 as in Construction 3.6 vanishes.*

*Proof.* It follows from Lemma 3.5, and the observation that $[z]$ is in the image of an element generating the source of $d^2$. $\square$

The strategy for the proof of the main theorem will be to choose $f$ in such a way that one can guarantee that, for every prime number $p$, the Toda bracket $[z]$ has higher $\mathrm{H}\mathbb{F}_p$-Adams filtration than any non-zero element in $\mathrm{coker}(J)_{8n-1}$. We will find such "$f$" in the $\infty$-category of synthetic spectra.

[2]The vanishing of the differentials $d^1 \circ d^1 = 0$ in the normalised chain complex is a consequence of the simplicial identity $d_0 d_0 = d_1 d_0$. The face maps of the bar construction satisfy this identity exactly because $J_+$ is a ring map.

### 3.4. **Synthetic spectra**

In this section, $E$ will always be an Adams-type homotopy associative ring spectrum, the kind of spectra for which Adams wrote the Künneth, universal coefficients, and Adams spectral sequences. To define what this means, let us start with the following.

**Definition 3.9.** Let $E$ be a homotopy associative ring spectrum. We say that a spectrum $X$ is *$E$-projective* if $E_*X$ is projective as a $\pi_*E$-module. We say that $X$ is *finite $E$-projective* if it is finite and $E_*X$ is finitely generated and projective over $\pi_*E$. We denote the $\infty$-category of finite $E$-projective spectra by $\mathcal{S}\mathrm{p}_E^{\mathrm{fp}}$.

For example, the spheres are finite $E$-projective for any choice of homology theory $E$. In fact, given a map of spectra $E \to E'$ realizing to one of algebras in the homotopy category, every (finite) $E$-projective spectrum is also (finite) $E'$-projective.

**Definition 3.10.** We say that a homotopy ring spectrum $E$ is *Adams-type* if it can be written as a filtered colimit $E \simeq \operatorname{colim} E_\alpha$ of finite $\mathbb{S}$-projective spectra such that for each $E_\alpha$ the natural map

$$E^*E_\alpha \longrightarrow \mathrm{Hom}_{\pi_*E}(E_*E_\alpha, \pi_*E)$$

is an isomorphism.

Let $E$ be an Adams-type ring spectrum. In [Pst18], a stable, presentable $\infty$-category $\mathcal{S}\mathrm{yn}_E$ is constructed, together with a functor

$$\nu_E \colon \mathcal{S}\mathrm{p} \to \mathcal{S}\mathrm{yn}_E,$$

called the *synthetic analogue* functor. Moreover, it admits a symmetric monoidal structure, preserving colimits in each variable, and promoting the restriction $\nu_E \colon \mathcal{S}\mathrm{p}_E^{\mathrm{fp}} \to \mathcal{S}\mathrm{yn}_E$ to a symmetric monoidal functor – see [Pst18, Proposition 4.2].

Nevertheless, for an arbitrary pair of spectra $X, Y$, the functor $\nu$ admits a lax symmetric monoidal structure, which gives a map

$$\nu_E X \otimes \nu_E Y \longrightarrow \nu_E(X \otimes Y).$$

In fact, this map is an equivalence when $X$ or $Y$ can be written as a filtered colimits of finite projectives [Pst18, Lemma 4.24]. We obtain the following important examples, and the only which we will make use of.

**Example 3.11.** For each prime number $p$, the ring spectrum $E = \mathrm{H}\mathbb{F}_p$ is of Adams-type, and every finite spectrum is finite $\mathrm{H}\mathbb{F}_p$-projective. Therefore, $\nu_{\mathrm{H}\mathbb{F}_p} \colon \mathcal{S}\mathrm{p} \to \mathcal{S}\mathrm{yn}_E$ is symmetric monoidal.

**Definition 3.12.** Let $t, w$ be integers. The *bigraded sphere* $\mathbb{S}^{t,w}$ is defined to be $\Sigma^{t-w} \nu_E \mathbb{S}^w$. For a synthetic spectrum $X$, the *bigraded homotopy groups* are defined to be the abelian groups

$$\pi_{t,w}(X) := \pi_0 \operatorname{Map}(\mathbb{S}^{t,w}, X).$$

Tensoring with the bigraded spheres defines autoequivalences $\Sigma^{t,w} := - \otimes S^{t,w} \colon \mathcal{S}\mathrm{yn}_E \to \mathcal{S}\mathrm{yn}_E$. This is due to the fact that, on the one hand, the suspension functor commutes with the cocontinuous tensor product and is an autoquivalence of the stable $\infty$-category $\mathcal{S}\mathrm{yn}_E$; on the other hand, tensoring with spheres defines an autoequivalence of $\mathcal{S}\mathrm{p}$, and $\nu_E$ underlies a symmetric monoidal functor when restricted to finite $E$-projective spectra.

We denote by $\tau$ the canonical limit comparison map

$$\tau \colon \nu_E(\Omega\mathbb{S}) \longrightarrow \Omega \nu_E(\mathbb{S}).$$

In terms of bigraded spheres, the suspension of $\tau$ takes the form $\tau \colon \mathbb{S}^{0,-1} \to \mathbb{S}^{0,0}$. We will need the following result to go from spectra to synthetic spectra, and back, where we say that a synthetic spectrum is *$\tau$-invertible* if $\tau \colon \Sigma^{0,-1} X \to X$ is an equivalence.

**Theorem 3.13** ([Pst18, Thm. 4.36, Prop. 4.39])**.** *The localization functor given by inverting $\tau$ admits a refinement to a symmetric monoidal functor. The full subcategory of $\tau$-invertible synthetic spectra is equivalent to the category of spectra. The composition of functors*

$$\mathcal{S}\mathrm{p} \xrightarrow{\nu_E} \mathcal{S}\mathrm{yn}_E \xrightarrow{\tau^{-1}} \mathcal{S}\mathrm{p}$$

*is equivalent to the identity functor.*

Next, we will present a first result relating "divisibility by $\tau$" of maps in $\mathcal{S}\mathrm{yn}_E$ and their $E$-Adams filtration is the one that follows. Notably, a more computationally precise result along these lines in [BHS19, Theorem 9.19].

**Definition 3.14.** The *$E$-Adams filtration* of a map of spectra $f : X \to Y$ is the minimum, if it exists, over the integers $k$ such that $0 \neq f \in F_E^k / F_E^{k+1}$, where $F_E^\bullet$ is the decreasing filtration of $\mathrm{Map}_{\mathcal{S}\mathrm{p}}(X, Y)$ constructed for the $E$-based Adams spectral sequence [Ada74, Theorem 5.1].

In fact, by analising the filtration, we have that a map has $E$-Adams filtration at least $k$ if it can be written as a composite of $k$-many maps which are zero on $E$-homology.

**Lemma 3.15** ([BHS19, Lemma 9.15])**.** *If a map of spectra $f : X \to Y$ has $E$-Adams filtration at least $k$, then there exists a factorisation*

$$\begin{tikzcd} & \Sigma^{0,k}\nu Y \arrow[d, "\tau^k"] \\ \nu X \arrow[ur, dashed] \arrow[r, "\nu_E f"'] & \nu Y. \end{tikzcd}$$

*in $\mathcal{S}\mathrm{yn}_E$.*

This gives one implication in the following.

**Theorem 3.16** ([BHS19][Corollary 9.21])**.** *Let $X$ be an $E$-nilpotent complete spectrum with strongly convergent $E$-based Adams spectral sequence. Then the filtration of $\pi_k(X)$ given by*

$$F^s\pi_k(X) := \mathrm{im}(\pi_{k,k+s}(\nu X) \xrightarrow{\tau^{-1}} \pi_k(X))$$

*coincides with the $E$-Adams filtration on $\pi_k(X)$.*

### 3.5. A synthetic Toda bracket

We finished Section 3.3 outlining the strategy for the proof of the main theorem, namely to bound the $\mathrm{H}\mathbb{F}_p$-Adams filtration of the Toda bracket associated to the data

$$\begin{tikzcd} \mathbb{S}^{8n-2} \arrow[r, "2(x\otimes x)"] \arrow[rr, bend left, "0"] \arrow[rr, phantom, bend left=25, "f\Updownarrow"] & \Sigma^\infty O\langle 4n-1\rangle^{\otimes 2} \arrow[r, "1\otimes J-m"] \arrow[rr, bend right, "0"'] \arrow[rr, phantom, bend right=25, "\Updownarrow a"] & \Sigma^\infty O\langle 4n-1\rangle \arrow[r, "J"] & \mathbb{S}, \end{tikzcd}$$

for every prime number $p$. More precisely, we want to choose a homotopy $f$ such that the resulting Toda bracket has high enough $\mathrm{H}\mathbb{F}_p$-Adams filtration – in the sense of Theorem 3.22 – to guarantee that it is trivial modulo the image of $J$.

Observe that it suffices to bound the filtration of a Toda bracket obtained by substituting the spectrum $\Sigma^\infty O\langle 4n-1\rangle$ by an $(8n-1)$-skeleton $M$, because the map $M \to \Sigma^\infty O\langle 4n-1\rangle$ is $8n$-connected. This replacement is justified by the following result.

**Lemma 3.17** ([BHS19, Lemma 10.4])**.** *Let $p$ be a prime. The $\mathrm{H}\mathbb{F}_p$-Adams filtration of the $p$-completed composite $M_p^\wedge \to \Sigma^\infty O\langle 4n-1\rangle_p^\wedge \to \mathbb{S}_p^\wedge$ is bounded below by*

$$N_p := \lfloor \frac{4n}{2p-2} \rfloor - \lfloor \log_p(4n) \rfloor$$

*for $p$ odd, and by*

$$N_2 := h(4n-1) - \lfloor \log_2(8n-1) \rfloor - 1$$

*for $p = 2$, where $h(k)$ is the number of positive integers less than or equal to $k$ which are congruent to $0, 1, 2$ or $4$ modulo $8$.*

**Convention 3.18.** In the remaining of this section, we fix a prime $p$ and all spectra are implicitly $p$-completed.

To replace $\Sigma^\infty O\langle 4n-1\rangle$ with its $(8n-1)$-skeleton in the Toda bracket, we must present it in a slightly different way because we can't make sense of $x^2$ in $M$, which no longer carries a ring structure. Consider instead

$$\begin{tikzcd}
\mathbb{S}^{8n-2} \arrow[r, "2"] \arrow[rr, bend left=50, "0"] \arrow[drr, bend right=40, "0"'] & \mathbb{S}^{8n-2} \arrow[r, "xJ(x)"] \arrow[dr, "J(x)^2"'] \arrow[u, phantom, "\Updownarrow f"] \arrow[d, phantom, "\Updownarrow h"] & M \arrow[d, "J"] \arrow[dl, phantom, "\Leftrightarrow g"] \\
& & \mathbb{S}.
\end{tikzcd} \tag{9}$$

We can similarly form a Toda bracket $\omega \in \pi_{8n-1}\mathbb{S}$, which is well defined after choosing homotopies issuing the commutativity of the above diagram. After choosing the canonical homotopies[3] $h\colon 2J(x)^2 \simeq 0$ and $g: J(xJ(x)) \simeq J(x)^2$, it is shown in [BHS19, Lemma 6.7] that the class of $\omega$ agrees with that of the previously defined Toda bracket $z$ in $\mathrm{coker}(J)_{8n-1}$.

We proceed by explaining how to lift the diagram 9 to the one below in $\mathcal{S}\mathrm{yn}_{\mathrm{H}\mathbb{F}_p}$, via the synthetic analogue functor $\nu_{\mathrm{H}\mathbb{F}_p}$, so that we can recover 9 by the $\tau$-inversion functor – cf. 3.13.

$$\begin{tikzcd}
\mathbb{S}^{8n-2,8n-2+2N_p} \arrow[r, "2"] \arrow[rr, bend left=50, "0"] \arrow[drr, bend right=40, "0"'] & \mathbb{S}^{8n-2},\mathbb{S}^{8n-2,8n-2+2N_p} \arrow[r, "\nu_{\mathrm{H}\mathbb{F}_p}(x)y"] \arrow[dr, "y^2"'] \arrow[u, phantom, "\Updownarrow \tilde{f}"] \arrow[d, phantom, "\Updownarrow \tilde{h}"] & \Sigma^{0,N_p}\nu_{\mathrm{H}\mathbb{F}_p}(M) \arrow[d, "\tilde{J}"] \arrow[dl, phantom, "\Leftrightarrow \tilde{g}"] \\
& & \mathbb{S}^{0,0}.
\end{tikzcd} \tag{10}$$

We start by lifting $x \in \pi_{4n-1}(M)$ to $\nu_{\mathrm{H}\mathbb{F}_p}(x) \in \pi_{4n-1,4n-1}\nu_{\mathrm{H}\mathbb{F}_p}(M)$. By 3.17 and 3.15, we can factor $\nu_{\mathrm{H}\mathbb{F}_p}(J)$ and obtain a map of synthetic spectra

$$\tilde{J} : \Sigma^{0,N_p}\nu_{\mathrm{H}\mathbb{F}_p}(M) \longrightarrow \mathbb{S}^{0,0}.$$

[3] By canonical, we mean the homotopies issuing the graded commutativity of the $\mathbb{E}_\infty$ ring structure on $\mathbb{S}$, and the fact that $J$ is a map of right $\mathbb{S}$-modules, respectively.

Since $\nu_{\mathrm{H}\mathbb{F}_p}$ underlies a symmetric monoidal functor, the commutative algebra structure on the sphere spectrum is transfered to one on its synthetic analogue $\mathbb{S}^{0,0}$; similarly we get a module structure on $\nu_{\mathrm{H}\mathbb{F}_p}(M)$ over $S^{0,0}$. Thus we can make sense of $y^2$, where we write $y := \tilde{J}\,\nu_{\mathrm{H}\mathbb{F}_p}(x)$; and of $\nu_{\mathrm{H}\mathbb{F}_p}(x)y$ in $\nu_{\mathrm{H}\mathbb{F}_p}(M)$.

Finally, the following result assures the existence of a nullhomotopy $\tilde{f}$ completing diagram 10, thus giving a Toda bracket $\tilde{\omega} \in \pi_{8n-2,8n-2+2N_p}\mathbb{S}^{0,0}$.

**Proposition 3.19** ([BHS19, Proposition 10.7]). *Let $n \geq 3$ and $s \geq 2$, then the bigraded homotopy groups $\pi_{8n-2,8n-2+s}\,\nu_{\mathrm{H}\mathbb{F}_p}(M_p^{\wedge})$ vanish. In particular, the group $\pi_{8n-2,8n-2+N_p}\,\nu_{\mathrm{H}\mathbb{F}_p}(M_p^{\wedge})$ vanishes.*

**Theorem 3.20** ([BHS19, Theorem 10.8]). *Let $p$ be a prime number. There exists a nullhomotopy $f$ fitting in diagram 9 which induces a Toda bracket $\omega$ with $H\mathbb{F}_p$-Adams filtration greater than or equal to $2N_p - 1$.*

*Proof.* Choose $f$ in 9 to be $\tau^{-1}(\tilde{f})$, for any nullhomotopy $\tilde{f}$ the existence of which is guaranteed by Proposition 3.19. Then, by Theorem 3.13, the resulting Toda bracket $\omega$ is the image under $\tau$-inversion functor of the synthetic Toda bracket $\tilde{\omega}$. In this situation, Theorem 3.16 gives that $\omega$ has $H\mathbb{F}_p$-Adams filtration greater than or equal to $(8n-2+2N_p)-(8n-1) = 2N_p - 1$. □

*Remark* 3.21. Observe that we have information about the Adams filtration of almost all maps in diagram 9. Indeed, we know that $J$ has Adams filtration at least $N_p$, and this is preserved by compositions, and doubled by products. But to establish the Adams filtration of the bracket we need to appropiately choose homotopies $f, g, h$ with high filtration. The ability to do so in synthetic spectra, where the Adams filtration of a map is recorded by its "divisibility by $\tau$", seems to be one of the key features of this theory. The other being that it avoids complications about the monoidality of the Adams filtration by not considering the monoidal structure on the filtration, but instead in the object mapping into the filtration. More precisely, the Adams spectral sequence targeting the homotopy groups of a spectrum $X$ is recovered in synthetic spectra by mapping $S^{0,0}$ into the tower over $\nu X$ given by multiplication by $\tau$.

### 3.6. Proof of the main theorem

We finish by referencing upper bounds for the $\mathrm{H}\mathbb{F}_p$-Adams filtration of elements outside the image $J$, and prove that they imply the main result. For a prime number $p > 2$, denote by $\Gamma_p^n$ the minimal integer $m$ such that every element $\alpha$ in the $p$-localisation $\pi_{8n-1}\mathbb{S}_{(p)}$ with $\mathrm{H}\mathbb{F}_p$-Adams filtration strictly greater than $m$ is in the image of $J$. Similarly, we write $\Gamma_2^n$ for the minimal integer $m$ such that every element $\alpha \in \pi_{8n-1}\mathbb{S}_{(2)}$ with $\mathrm{H}\mathbb{F}_2$-Adams filtration strictly greater than $m$ is in the subgroup generated by the image of $J$ and the $\mu$-family.

**Theorem 3.22.** *Let $p$ be a prime number and $n$ a positive integer. The following expressions are upper bounds for $\Gamma_p^n$ :*

- [DM89, Corollary 1.3] *If $p = 2$, then $\Gamma_p^n < 3 + \nu_2(n)$, where $\nu_2(k)$ is the 2-adic valuation of $k$.*
- [BHS19, Theorem B7] *If $p = 3$, then*

$$\Gamma_p^n < \frac{25(8n-1)}{184} + 19 + \frac{1133}{1472}.$$

- [Gon00, Theorem 5.1] *If $p \geq 3$, then*

$$\Gamma_p^n < 3 + \frac{(2p-1)(8n-1)}{(2p-2)(p^2-p-1)}.$$

*Proof.* These follow from the cited references by [BHS19, Remark 7.9] and the fact that $\pi_{8n-1}\mathbb{S}_{(p)}$ splits as a direct sum of the image of $J$ and the kernel of the $K(1)$-local Hurewicz morphism. □

Finally, we can conclude the proof of the main theorem.

**Theorem 3.23** ([BHS19])**.** *For all natural numbers $n > 31$, the kernel of the unit map*

$$\pi_{8n-1}\mathbb{S} \to \pi_{8n-1}\,\mathrm{MO}\langle 4n\rangle$$

*is the image of the $J$-homomorphism.*

*Proof.* By Lemma 3.5, it is enough to see that the referenced differential vanishes. By Lemma 3.8, this will follow if the Toda bracket $[z] \in \mathrm{coker}(J)_{8n-1}$ vanishes. As mentioned before, this Toda bracket identifies with $\omega$ by [BHS19, Lemma 6.7].

Finally, it suffices to show that if $n > 31$, then for every prime number $p$ the $\mathrm{H}\mathbb{F}_p$-Adams filtration of $\omega$ is greater than $\Gamma_p^n$. This is indeed also true for $p = 2$ because according to [BHS19, Remark 7.4], the elements in the $\mu$-family do not occur in dimension $8n-1$. Note that, by Theorem 3.20, the Toda bracket $\omega$ has $\mathrm{H}\mathbb{F}_p$-Adams filtration at least $2N_p - 1$.

It is shown in [BHS19, Lemma 7.15] that $2N_p - 1$ is strictly bigger than all the bounds in Theorem 3.22 if $p < 11$ and $n > 31$. Now fix $n > 31$, and let $p \geq 11$ be arbitrary. If $4p - 4 \leq n$, then $2N_p - 1$ is strictly bigger than the bounds of Theorem 3.22. Otherwise, one can show by hand that there are no non-zero elements in $\mathrm{coker}(J)_{(p)}$ of dimension $8n-1$ [4] – see the proof of [BHS19, Theorem 7.1]. □

## References


[Ada74] John Frank Adams. *Stable homotopy and generalised homology*. University of Chicago press, 1974.

[And+14] Matthew Ando et al. "Units of ring spectra, orientations, and Thom spectra via rigid infinite loop space theory". In: *Journal of Topology* 7.4 (2014), pp. 1077–1117.

[Ati66] M. F. Atiyah. "Thom complexes". In: *Matematika* 10.5 (1966), pp. 48–69.

[BHS19] Robert Burklund, Jeremy Hahn, and Andrew Senger. "On the boundaries of highly connected, almost closed manifolds". In: *arXiv: Algebraic Topology* (2019).

[DM89] Donald M. Davis and Mark Mahowald. "The image of the stable J-homomorphism". In: *Topology* 28.1 (1989), pp. 39–58.

[Elm+97] AD Elmendorf et al. *Rings, modules, and algebras in stable homotopy theory*. Vol. 47. 1997.

[GR16] Søren Galatius and Oscar Randal-Williams. "Abelian quotients of mapping class groups of highly connected manifolds". In: *Mathematische Annalen* 365 (June 2016).

[Gon00] Jesús González. "Odd Primary bo Resolutions and Classification of the Stable Summands of Stunted Lens Spaces". In: *Transactions of the American Mathematical Society* 352.3 (2000), pp. 1149–1169.

[KM63] Michel A. Kervaire and John W. Milnor. "Groups of Homotopy Spheres: I". In: *Annals of Mathematics* 77.3 (1963), pp. 504–537.


[4] More precisely, for $p = 11, 13$ the known calculations of $\mathrm{coker}(J)_{(p)}$ [Rav86] inform us that there is no element of dimension $8n-1$; and that for $p \geq 17$, the non-zero elements in $\mathrm{coker}(J)_{(p)}$ appear in dimension $2p^2 - 2p - 2 > 8(4n-4) - 1 \geq 8n - 1$.

[Pst18] Piotr Pstrągowski. *Synthetic spectra and the cellular motivic category*. 2018. arXiv: 1803.01804 [math.AT].

[Rav86] Douglas C. Ravenel. "Complex Cobordism and Stable Homotopy Groups of Spheres". In: 1986.

[Sto85] Stephan Stolz. *Hochzusammenhängende Mannigfaltigkeiten und ihre Ränder*. Springer Berlin Heidelberg, 1985.

[Wal67] C. T. C. Wall. "Classification problems in differential topology—VI: Classification of ($s$1)-connected ($2s+1$)-manifolds". In: *Topology* 6 (1967), pp. 273–296.