Synthetic Spectra I

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Abstract

These are expository notes on the topic of synthetic spectra for the 2021 MSc course Topics in Algebraic Topology at the University of Copenhagen. The goal of this notes is to setup the thoery of synthetic spectra and its main properties. This is the first part of a two-part exposition with Jan McGarry. These notes roughly follow [Pst18]. Any suggestions and/or corrections are welcome.

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1 Introduction

The starting point for the study of synthetic spectra is the classical Adams spectral sequence. One of the central problems of (stable) homotopy theory is the computation of homotopy groups of spectra. The Adams spectral sequence introduced by Adams in [Ada74] has been the most successfull tool for attacking this problem. Let us start by defining this spectral sequence.

Theorem 1.1. [Ada74] Let E be an Adams-type ring spectrum and Y a finite E-projective spectrum. If X is an arbitrary spectrum, there exists a convergent spectral sequence

$$\mathrm{E}^2_{s,t} \simeq \mathrm{Ext}^{-s,t}_{E_*E}(E_*Y, E_*X) \Rightarrow \pi_{s+t}\,\mathrm{map}(Y, X^\wedge_E)$$

Let p be a prime. If we specify this result to $E = \mathrm{H}\mathbb{F}_p$, $Y = \mathbb{S}$ and X finite connective, we have the following sequence

$$\mathrm{E}^2_{s,t} \simeq \mathrm{Ext}^{-s,t}_{\mathscr{A}_p}(\mathbb{F}_p, H_*(X, \mathbb{F}_p)) \Rightarrow \pi_{t+s}(X^\wedge_p)$$

converging to the p-adic completion of the homotopy groups of X. On another hand, we can also define the p—Bockstein spectral sequence for X. We start with the filtration

$$\cdots \xrightarrow{p} X \xrightarrow{p} X \xrightarrow{p} X \xrightarrow{p} \cdots$$

where $X \xrightarrow{p} X$ is the scalar multiplication map arising from $\mathbb{S} \xrightarrow{p} \mathbb{S}$. Therefore we get the spectral sequence

$$\mathrm{E}^1_{s,t} \simeq \pi_{t+s}(X/p) \Rightarrow \pi_{t+s}(X^\wedge_p)[p^{-1}].$$

The nature of these spectral sequences is quite different from each other. Let us briefly compare them:

1. One can argue that the Bockstein spectral sequence arises in a more natural way, for we can produce the Bockstein filtration independently of X. This could imply that we might be able to extract more information on its differentials. On the other hand, the Adams filtration arises somewhat more misteriously and therefore extracting information on the differentials is a running problem of this sequence.

2. However, one can also argue that the Adams spectral sequence is computationally more advantageous, for its E_2 page is purely algebraic, modulo computing the homology of X. The E_2 is usually called the Hopf algebra cohomology of $\mathscr{A}_p$ with coefficients in $H_*(X, \mathbb{F}_p)$. One usually calculates this groups with (relative) injective resolutions or using the May spectral sequence. On the other hand, the homotopy groups of $X/p := \mathrm{cof}(p : X \to X)$ are not easier than the ones of X, therefore this sequence is computationally disadvantageous.

On yet another perspective, the category of spectra has a natural t-structure given by connective and coconnective spectra. This produces the Postnikov tower

$$\cdots \to \tau_{\geq n+1}X \to \tau_{\geq n}X \to \tau_{\geq n-1}X \to \cdots$$

for every spectrum X. Therefore, we have the (trivial) Postnikov spectral sequence

$$\mathrm{E}^1_{s,t} \simeq \pi_{s+t}(\Sigma^s \mathrm{H}\pi_s X) \Rightarrow \pi_{s+t}X.$$

At first sight, this spectral sequence might appear useless and in fact, at least computationally, it is. However, morally there is an important feature of the Postnikov tower: its associated graded lies (up to suspensions) in the heart of **Sp**, which can be canonically identified with **Ab**. The uselessness of this sequence lies on the fact that the notion of homotopy group intrinsic to the category of spectra is the same as the notion arising from its t-structure. This will not be the case in synthetic spectra. In this case, the "intrinsic" homotopy groups of objects in the heart will be naturally identified with the Ext groups in the Adams spectral sequence.

One of the biproducts of the theory of synthetic spectra is that it provides a context where the Adams spectral sequence of X arises as both a Bockstein spectral sequence and a Postnikov spectral sequence and then hopefully unifying both perspectives. We will pick up this topic with some applications in the next subsection.

1.1 Overview of the theory

In this section, we will try to overview the theory of synthetic spectra and present some examples. In the end, we hope to touch on some applications of the synthetic deformation to study the classical Adams spectral sequence.

We start with a (sufficiently commutative, *e.g.* $\mathbb{E}_1$) multiplicative cohomology theory E. For such an E satifying some flatness conditions (*e.g.* Adams-type), the homology $E_*E = \pi_*(E^{\otimes[1]})$

can be promoted to a Hopf algebroid in a prefered manner, which generalizes the case for $E = H\mathbb{F}_2$ where we have a Hopf algebra. Goerss and Hopkins proved that the suitable category of (co)modules over such Hopf algebroid can be generated by suitable compact projectives by endowing this subcategory with a topology and taking product preserving sheaves. The category of product preserving presheaves should be thought of as a completion of the indexing category with respect to filtered colimits. The sheaf condition should account for exactness properties.

In order to lift this story to a homotopy theoretical setting, one should replace compact projective comodules by compact E-projective spectra. We arrive at the definition of Syn_E as product preserving sheaves of spectra in the category of suitable compact E-projective spectra. Intuitively, this should be thought of as deriving the category of spectra with respect to the cohomology theory E where we resolve an arbitrary spectrum by its Adams filtration.

The main feature of Syn_E is that it is a one-parameter deformation of the category of spectra. More precisely, there exists a map $\tau : \mathbb{S}^{0,-1} \to \mathbb{S}^{0,0}$ such that we can identify $\mathbf{Sp}$ as the subcategory of τ-local objects $\mathrm{Syn}_E[\tau^{-1}]$ with help of the spectral Yoneda embedding. However, we can also embed the category of spectra in synthetic spectra by the synthetic analogue functor

$$\nu : \mathbf{Sp} \to \mathrm{Syn}_E$$

such that the Bockstein spectral sequence with respect to τ for νX can be identified with the Adams spectral sequence of the spectrum X.

The map τ is a map of bigraded spheres, which are topological and weighted suspensions of the synthetic analogue of the sphere spectrum. These are similar in nature to the bigraded spheres arising in the motivic deformation of $\mathbf{Sp}$. One important step to recover the Adams spectral sequence is the following identification[1].

$$[\mathbb{S}^{s+t,t}, \mathrm{cof}\,\tau \otimes X] \simeq \mathrm{Ext}^{-s,t}_{E_*E}(\pi_* E, E_* X)$$

We finish with some properties of the synthetic deformation. Firstly, we can form the (purely formal) following stable recollement associated to the idempotent algebra $\mathbb{S}^{0,0}$. Intuitively, this can thought of as a "split short exact sequence" of stable ∞-categories. We will not pursue this point of view.

$$\mathrm{Syn}_E^{\tau-nil} \underset{j_*}{\overset{j_!}{\underset{\longleftarrow}{\overset{j^* \simeq j^!}{\rightleftarrows}}}} \mathrm{Syn}_E \underset{i^!}{\overset{i^*}{\underset{\longleftarrow}{\overset{i_*}{\rightleftarrows}}}} \mathrm{Syn}_E[\tau^{-1}] \simeq \mathbf{Sp}$$

Moreover, we call the functor i^*, τ inversion and denote it by τ^{-1}. On another hand, Syn_E has a preferred t-strucute such that the functor ν can be expressed as the composition $\tau_{\geq 0} \circ i_* \circ Y$ where Y is the spectral Yoneda embedding. Finally, we can identify the τ-filtration quotients as elements in $\mathbf{Stable}_{E_*E}$, Hovey's derived category of comodules. This category should encodes the algebraic data of the Adams spectral sequence. Moreover, if we restrict to the essential image of ν, the filtration quotient lies in objects concentrated in degree 0, i.e. in $\mathbf{Comod}_{E_*E}$.

1.1.1 Some examples

This section will present some examples of results on the structure of Syn_E for some familiar (co)homology theories. In general it is very difficult to grasp the structure of Syn_E so we will only focus on the extreme cases, the initial $\mathbb{S}$ and the "final" MU.

[1]In this formula, the Ext-group $\mathrm{Ext}^{s,t}_{E_*E}(A,B)$ of two graded E_*E-comodules A and B is the t-graded part of the s-th right derived functor of $\mathrm{Hom}(A,-)$ evaluated at B

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Let us start with the (trivial example of the) sphere spectrum representing homotopy and cohomotopy theories. This spectrum is Adams-type by [Ada74, Chapter III.13.3]. One can see that the Hopf algebroid associated to $\mathbb{S}_*\mathbb{S}$ is discrete[2] which implies that $\mathbf{Comod}_{\mathbb{S}_*\mathbb{S}} \simeq \mathbf{Mod}_{\pi_*\mathbb{S}}$. Let us start by identifying $\mathbf{Mod}^{\mathrm{fp}}_{\pi_*\mathbb{S}}$.

Claim 1. Any finite projetive $\pi_*\mathbb{S}$-module is free.

Proof. This is a special case of [Lin71, Theorem 1], which states that locally finitely generated $\pi_*\mathbb{S}$-modules with finite projective dimension are free. □

Therefore, it follows that any finite $\mathbb{S}$-projective spectrum is equivalent to a wedge of spheres. The product preserving condition implies that a synthetic spectrum is determined by its values on spheres. Using a similar argument as in [Pst18, Theorem 6.2], we see that Syn_E is generated under colimits by synthetic analogues of wedges of spheres. By [Pst18, Lemma 4.23], we see that Syn_E is generated by synthetic analogues of spheres, that is bigraded spheres. This property is called cellularity.

This example is (almost) worthless since the $\mathbb{S}$—based Adams spectral sequence collapses in the second page. This is true for any cohomology theory where π_*E is acyclic as a E_*E-comodule. This behaviour suggests the following result.

Claim 2. The map τ is an equivalence. Therefore, the spectral Yoneda embedding $Y : \mathbf{Sp} \to \mathrm{Syn}_{\mathbb{S}}$ is an equivalence.

Proof. This follows from the identification of the homotopy groups of $\mathrm{cof}\,\tau$ as the E_2 page of the Adams spectral sequence which collapses. This implies that $\mathrm{cof}\,\tau$ is equivalent to 0. □

Cellularity recovers the fact that $\mathbb{S}$ generates $\mathbf{Sp}$ under colimits and shifts.

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Let us finish by mentioning the (non-trivial) identification of $\mathrm{Syn}_{\mathrm{MU}}$ due to [Pst18] where MU is the complex cobordism spectrum. Similarly to the previous case, any finitely generated projective module over $\pi_*(\mathrm{MU})$ is free [Pst18, Lemma 6.1]. We have the following result.

Theorem 1.2. [Pst18, Theorem 6.2] The category $\mathrm{Syn}_{\mathrm{MU}}$ is cellular in the sense that it is generated under colimits by the bigraded spheres.

To conclude, we present the main result in [Pst18]. Consider the subcategory $\mathbf{Sp}_E^{fpev}$ of finite spectra with E-homology finitely generated projective and concentrated in even degrees. Then we can produce even synthetic spectra as the category $\mathrm{Syn}_E^{ev} := \mathrm{Sh}_\Sigma(\mathbf{Sp}_E^{fpev}, \mathbf{Sp})$, which has very similar properties. Denote by $\mathbf{Sp}_{\mathbb{C}}$ the cellular motivic category which is defined as the smallest subcategory of complex motivic spectra closed under colimits and containing bigraded spheres. Then we have the following theorem which is the main application of the synthetic technology. We will see in the next section an example of one of many applications of this theory. We expect this theory to have a plethora of future applications.

Theorem 1.3. [Pst18, Theorem 1.4] There exists an adjoint pair $\Theta_* : \mathbf{Sp}_{\mathbb{C}} \leftrightarrows \mathrm{Syn}_{\mathrm{MU}}^{ev} : \Theta^*$ which induces an adjoint equivalence after p-completion.

[2] This holds whenever E is an idempotent ring spectrum, which is clearly true for $\mathbb{S}$.

1.1.2 Applications to Adams filtrations

In this short section, we will present some concrete applications of the synthetic deformation to Adams filtrations of maps. These results are due to [BHS19, Section 9]. Let X be E-nilpotent complete spectrum with strongly convergent E-Adams spectral sequence. Recall that a map $x : \mathbb{S}^t \to X$ has E-Adams filtration s if it can be written as a composition of s maps which are zero in E-homology. This is one form of the Adams filtration on $\pi_* X$. Since $\tau^{-1} \nu X = X$, τ-inversion gives a family of maps $\pi_{t,t+s} \nu X \to \pi_t X$. We will focus on the following two results. The first result makes precise the claim that τ-divisibility of a map coincides with its Adams filtration.

Proposition 1.1. [BHS19, Corollary 9.21] Let $F^* \pi_t X$ be the filtration of $\pi_* X$ given by

$$F^s \pi_t X := \mathrm{im}(\pi_{t,t+s} \nu X \to \pi_t X).$$

In this situation, $F^* \pi_t X$ coincides with the Adams filtration.

Proposition 1.2. [BHS19, Corollary 9.20] If a, b are integers such that $\pi_{a,b+s}(\mathrm{cof}\, \tau \otimes \nu X)$ vanishes for all non-negative s. Then, it follows that

$$\pi_{a,b+s} \nu X \simeq 0$$

for all $s \geq 0$.

These two results combined provide a vanishing condition for certain high filtration elements of the homotopy groups of X based on the vanishing of a certain line in the E_2 page of its Adams spectral sequence.

Corolary 1.1. If a, b are integers such that $\pi_{a,b+s}(\mathrm{cof}\, \tau \otimes \nu X) \simeq \mathrm{E}^2_{a-b-s,b+s}(X)$ vanishes for all non-negative s. Then, the k-th Adams filtration step $\mathrm{AF}^k \pi_a X \subset \pi_a X$ vanishes for every $k \geq a - b$.

1.2 Hopf algebroids and its comodules

We start by studying Hopf algebroids and its associated abelian category of comodules. Recall that, intuitively, a Hopf algebra over a field k is an associative k-algebra H together with a diagonal map $\Delta : A \to A \otimes A$ making (H, Δ) into a coassociative counital coalgebra such that both operations are compatible.

Remark 1.1. Let H be a Hopf algebra over k with diagonal map Δ. For any other commutative algebra A over k, we have that coassociativity and counitality induce a group structure on $\hom(H, A)$ given by $\hom(H, A) \otimes \hom(H, A) \to \hom(H \otimes H, A) \to \hom(H, A)$. Therefore, the Hopf algebra structure determines a lift of the corepresentable functor $\hom(H, -) : \mathbf{CAlg}_k \to \mathbf{Set}$ to **Groups**. Formally, this lifting determines a cogroup object in the category of commutative algebras. Moreover, this is equivalent to the structure of an affine group scheme $\mathrm{Spec}(H)$.

Definition 1.1. Let R be a (graded) commutative ring. A Hopf algebroid over R is a cogroupoid object in the category of commutative algebras over R.

To unpack this definition, let us recall a general definition of a groupoid object in a category $\mathbf{C}$. A groupoid object in $\mathbf{C}$ is a functor $X : \Delta^{op} \to \mathbf{C}$ such that for every n and $S, S' \subset [n]$ such that $S \cup S' = [n]$ and $S \cap S' = \{s\}$ for some $s \in [n]$, the following diagram is cartesian.

$$\begin{array}{ccc} X([n]) & \longrightarrow & X(S') \\ \downarrow & & \downarrow \\ X(S) & \longrightarrow & X(\{s\}) \end{array}$$

We can start by recovering the source and target maps as d^1 and d^0. On the other hand, the decomposition $\{0,1\} \cup \{1,2\}$ induces an isomorphism $X_2 \to X_1 \times_{X_0} X_1$, provided that $\mathbf{C}$ admits pullbacks. We can define the composition map $X_1 \times_{X_0} X_1 \to X_1$ by first using the inverse of the above isomorphism and post-composing it with d^1. One can observe that the condition above implies that X_0 and X_1 along with unit, source, target, inverse and composition map determine the groupoid and therefore we will follow the tradition in Homotopy theory and abbreviate it as (X_0, X_1).

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Dually, a cogroupoid object is a functor $X : \Delta \to \mathbf{C}$ satisfying dual conditions. Let us now present how to get a Hopf algebroid from a homotopy commutative ring spectrum E. Define the following cosimplicial object in the category of commutative algebras in graded abelian groups,

$$X : \Delta \to \mathbf{CALg}(\mathbf{grAb})$$
$$[n] \mapsto \pi_*(E^{\otimes[n]})$$

where $E^{\otimes[n]}$ is just the $(n+1)$—fold product of E remembering the ordering. Note that $X_0 = \pi_* E$ and $X_1 = E_* E$. In order for this cosimplicial object to be a groupoid, the map $d^2 \cdot d^0 : E_* E \otimes_{\pi_1 E} E_* E \to \pi_*(E \otimes E \otimes E)$ is an isomorphism. This is not true in general. However, in the following Tor spectral sequence.

$$\mathrm{E}^2_{s,t} = \mathrm{Tor}^{\pi_* E}_{s,t}(E_* E, E_* E) \Rightarrow \pi_*(E \otimes E \otimes E)$$

this is always the edge morphism and if E is Adams type, it is an isomorphism.

This sequence collapses since $E_* E$ is flat as a $\pi_* E$-module[3]. The next proposition shows that this is a sufficient condition for this cosimplicial object to be a groupoid. We will abbreviate this groupoid by $(\pi_* E, E_* E)$.

Proposition 1.3. Let E be a homotopy commutative ring spectrum such that $E_* E$ is flat over $\pi_* E$. In this situation, $(\pi_* E, E_* E)$ is a Hopf algebroid over $\pi_* E$.

The main examples of such ring spectra are Adams-type ring spectra which include $H\mathbb{F}_p$ for any prime p and MU. The first example gives a Hopf algebra since the source and target maps are the same, making it into a group object. This follows from the fact that there is a unique non-trivial algebra map from $\pi_* H\mathbb{F}_p \simeq \mathbb{F}_p$ to $H\mathbb{F}_{p*} H\mathbb{F}_p$. The same is not true for MU, which motivated the definition of Hopf algebroid. This is the content of the Landweber-Novikov theorem which identifies the Hopf algebroid structure of $(\pi_* \mathrm{MU}, \mathrm{MU}_* \mathrm{MU})$. For the details, see [Rav86, 4.1.11 and A2.1.16].

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We can then define comodules over a Hopf algebroid and morphisms between them.

Definition 1.2. Let (A, Γ) a Hopf algebroid. A comodule over (A, Γ) is a module M over A along with a Γ-linear map map

$$\epsilon : d^{1^*} M \to d^{0*} M$$

satisfying the unit and cocycle conditions, where d^{j*} is the extension of scalars with respect to the map $d^j : A \to \Gamma$. A morphism of comodules is a morphism of A-modules commuting with the respective counit maps. Denote by $\mathbf{Comod}_\Gamma$ the category of comodules over (A, Γ).

[3]Here we consider $E_* E$ as a right $\pi_* E$-module given by the map d^1 which corresponds to the induced map of the unit $\mathbb{S} \to E$ in E_*-homology.

Remark 1.2. The notion of Γ-linear map $\epsilon : d^{1^*}M \to d^{0*}M$ is equivalent to the data of an A-linear map $\epsilon : M \to d^1_* d^{0*}M \simeq \Gamma \otimes M$ satisfying appropriate conditions. Here we consider Γ has the left A-module structure and its right A-module structure induces a module structure on $\Gamma \otimes M$.

Let (M, ϵ) and (N, μ) be Γ-comodules, we define its tensor product $(M, \epsilon) \otimes (N, \mu)$ as the comodule $(M \otimes_A N, \epsilon \otimes \mu)$, where [4]

$$\epsilon \otimes \mu : d^{1*}(M \otimes_A N) \simeq d^{1*}M \otimes_\Gamma d^{1*}N \overset{\epsilon \otimes_\Gamma \mu}{\to} d^{0*}M \otimes_\Gamma d^{0*}N \simeq d^{0*}(M \otimes_A N).$$

The following proposition summarizes good categorical properties for the category of comodules for a flat Hopf algebroid.

Proposition 1.4 ([Hov03]). The category $(\mathbf{Comod}_\Gamma, \otimes)$ is (co)complete closed symmetric monoidal abelian category, provided that Γ is a flat A-module. Moreover, given a morphism of Hopf algebroid $\phi : (A, \Gamma) \to (B, \Sigma)$ induces a symmetric monoidal functor $\phi_* : \mathbf{Comod}_\Gamma \to \mathbf{Comod}_\Sigma$ which admits a right adjoint ϕ^*.

The first results that sets this category apart from ordinary abelian categories like module categories is the following. Define the subcategory $\mathbf{Comod}_\Gamma^{\mathrm{fp}}$ of finitely generated projective comodules.

Theorem 1.4. If (A, Γ) is a flat Hopf algebroid, then the Yoneda embedding

$$y : \mathbf{Comod}_\Gamma \to \mathrm{Sh}_\Sigma(\mathbf{Comod}_\Gamma^{\mathrm{fp}}, \mathbf{Set})$$

is an equivalence.

Remark 1.3. This property holds if we replace $\mathbf{Comod}_\Gamma$ with any compactly generated Grothendieck abelian category and $\mathbf{Comod}_\Gamma^{\mathrm{fp}}$ by a choice of compact generators. See further in section 2.5 in [Pst18].

1.2.1 The derived category of stable comodules

In this section, we will briefly mention a way to construct a derived category of comodules over a Hopf algebroids which aims to understand them as homotopy theoretic objects. Recall that to any abelian category $\mathbf{A}$, we can associate its derived category $\mathscr{D}(\mathbf{A})$ constructed by inverting quasi-isomorphisms in the category of chain complexes in $\mathbf{A}$. Let (A, Γ) be in the conditions of 1.4. In [Hov03], Hovey argues that $\mathscr{D}(\mathbf{A})$ doesn't encode all the expected homotopical information. To fix this, Hovey introduces a different notion of weak equivalence. Hovey starts by fixing an injective resolution LA of A in $\mathbf{Comod}_\Gamma$.

Definition 1.3. Let $M \in \mathbf{Comod}_\Gamma$ be a simple comodule (i.e. no nontrivial proper submodules) and X be an arbitrary chain complex of comodules. We define the n-th homotopy group of X with respect to M as $\pi_n^M X := [M[n], LA \otimes X]$ as the homotopy classes of maps between chain complexes.

We say that a map of chain complexes $f : X \to Y$ is a weak equivalence if it induces an isomorphism of all homotopy groups for all $n \in \mathbb{Z}$ and M simple. We define $\mathbf{Stable}_\Gamma$ as the localization of unbounded chain complexes with respect to the previously defined weak equivalences. Under some additional conditions (which will be satisfied when E is of Adams type), we have the following result.

[4] One can show that flatness of d^1 implies the flatness of d^0 and therefore both d^{0*} and d^{1*} are symmetric monoidal.

Theorem 1.5. [Pst18, Theorem 3.7] If (A,Γ) is an Adams Hopf algebroid, then the Yoneda embedding $\mathbf{Stable}_\Gamma \to \mathrm{Sh}_\Sigma(\mathbf{Comod}_\Gamma^{\mathrm{fp}}, \mathbf{Sp})$ is an equivalence.

We will see that this implies that the heart of the natural t-structure coming from chain complexes is equivalent to $\mathbf{Comod}_\Gamma$. This category recovers the information of the Adams spectral sequence in the following way.

Proposition 1.5. Let X, Y be comodules viewed as chain complexes concentrated in degree 0. In this situation, there exists a canonical equivalence

$$[X,Y]_n := \pi_n(\mathrm{map}_{\mathbf{Stable}_\Gamma}(X,Y)) \simeq \pi_n(\mathrm{map}_{\mathscr{D}(\mathbf{Comod}_\Gamma)}(X,Y)) \simeq \mathrm{Ext}_\Gamma^{-n}(X,Y).$$

2 Synthetic spectra

Recall that 1.4, gives us that the Yoneda embedding $\mathbf{Comod}_\Gamma \to \mathrm{Sh}_\Sigma(\mathbf{Comod}_\Gamma^{\mathrm{fp}}, \mathbf{Set})$ is an equivalence and that by 1.5 we have an equivalence $\mathbf{Stable}_\Gamma \simeq \mathrm{Sh}_\Sigma(\mathbf{Comod}_\Gamma^{\mathrm{fp}}, \mathbf{Sp})$. On the other hand, we will see in 2.3 that the E-homology functor $E_* : \mathbf{Sp}_E^{\mathrm{fp}} \to \mathbf{Comod}_{E_*E}$ induces an equivalence on spherical sheaves of sets. Motivated by this results, we define synthetic spectra in the following way.

Definition 2.1. Let E be a Adams-type ring spectrum. The ∞-category of synthetic spectra is $\mathrm{Syn}_E := \mathrm{Sh}_\Sigma(\mathbf{Sp}_E^{\mathrm{fp}}, \mathbf{Sp})$ where $\mathbf{Sp}_E^{\mathrm{fp}}$ is the category of finite E-projective spectra, that is finite spectra X such that E_*X is projective as a π_*E-module.

Remark 2.1. This definition depends on the definition of the Grothendieck topology on $\mathbf{Sp}_E^{\mathrm{fp}}$ defined in 2.5.

We will see that we can embed the category of spectra in the category of synthetic spectra associated to E via the construction of the synthetic analogue of a spectrum. We construct it as follows. Let X be a spectrum, then notice that $y(X) := \mathrm{hom}_{\mathbf{Sp}}(-, X)$ is a product preserving presheaf of spaces. We can define the synthetic analogue νX of X as $\Sigma_+^\infty y(X)$, where $y : \mathbf{Sp} \to \mathrm{Sh}_\Sigma(\mathbf{Sp}_E^{\mathrm{fp}})$ and Σ_+^∞ is the left adjoint to Ω^∞. The remainder of this section is devoted to study categorical properties of Syn_E and the functor $\nu : \mathbf{Sp} \to \mathrm{Syn}_E$. This requires us to study the more general contexts of ∞-categories of spherical spherical sheaves on certain ∞-sites.

2.1 Spherical sheaves of anima and spectra

In this section, we will discuss the formal categorical properties of a general category of spherical sheaves. This category will not arrive as a category of sheaves, but we will prove various results that show that this category mimics the behaviour of such categories.

Remark 2.2. We will start by establishing these "hygenic" properties and structures for categories of sheaves of anima. We will see that considering the analog of sheaves of spectra (which specifies to our context) is a model for the stabilization of the above category. Therefore most of these properties will be preserved and structures will be chosen canonically in sheaves of spectra.

2.1.1 Main properties of spherical sheaves

Let $\mathbf{C}$ be an additive $\infty-$category, i.e. it admits finite products and finite coproducts, they agree and its homotopy category is additive. Denote by $\mathrm{PSh}_\Sigma(\mathbf{C})$ the ∞-category of spherical (i.e. product preserving) presheaves of anima. By [Lur17, Propostion 2.4.5.5] , this category is additive.

In order to define spherical sheaves, one should now consider a particular type of ∞-sites where the underlying additive structure is compatible with the covering data. This means that we should expect the correspondent functor of sheafification to preserve the sphericity of the presheaf. In this spirit, we define the following notion.

Definition 2.2. An additive ∞-site is a small ∞-site $\mathbf{C}$ which is additive as an ∞−category such that every covering sieve is generated by a single map.

The first advantage of this definition lies in the following result.

Proposition 2.1. [Pst18, Propositions 2.5 and 2.6] Let $\mathbf{C}$ be an additive ∞−site. In this situation, the sheafification functor $L : \mathrm{PSh}(\mathbf{C}) \to \mathrm{Sh}(\mathbf{C})$ takes spherical presheaves to spherical sheaves. Moreover, we can consider its restriction of L to $\mathrm{PSh}_\Sigma(\mathbf{C}) \to \mathrm{Sh}_\Sigma(\mathbf{C})$ and it is an acessible left exact localization. In particular, $\mathrm{Sh}_\Sigma(\mathbf{C})$ is presentable.

We have a recognition principle for spherical spectra that allows us to prove that $\mathrm{Sh}_\Sigma(\mathbf{C})$ is closed under filtered colimits as a subcategory of $\mathrm{PSh}_\Sigma(\mathbf{C})$.

Theorem 2.1. [Pst18, Theorem 2.8] Let $\mathbf{C}$ be an additive ∞-site. If $X \in \mathrm{PSh}_\Sigma(\mathbf{C})$, then the following are equivalent:

1. X is a spherical sheaf,
2. For every fiber sequence $F \to B \to A$ where $B \to A$ is a generated by a covering sieve, then $X(A) \to X(B) \to X(F)$ is a fiber sequence.

Remark 2.3. When we consider $\mathbf{C} = \mathbf{Comod}_\Gamma$ with the epimorphism topology, the second condition amounts to asking that X takes short exact sequences to fiber sequences.

An immediate corolary of the previous theorem is the following.

Corolary 2.1. The full subcategory $\mathrm{Sh}_\Sigma(\mathbf{C}) \subset \mathrm{PSh}_\Sigma(\mathbf{C})$ is closed under filtered colimits, therefore they are computed pointwise.

Proof. This follows immediatly from the recognition principle and the fact that the filtered colimits commute with finite limits and in particular commute with fiber sequences. □

2.1.2 Symmetric monoidal structures

It turns that in the context of synthetic spectra, there is a natural symmetric monoidal structure on Syn_E. This will be more general in the context of a particular type of ∞-site that is compatible with an underlying symmetric monoidal structure.

Definition 2.3. An excellent ∞−site is a symmetric monoidal category $\mathbf{C}^\otimes$ admitting duals along with a topology on $\mathbf{C} := \mathbf{C}^\otimes_{\langle 1 \rangle}$ making it into an additive ∞-site such that the functor $c \otimes - : \mathbf{C} \to \mathbf{C}$ takes coverings to coverings for every $c \in \mathbf{C}$.

Remark 2.4. Once again this property will be verified for $\mathbf{C} = \mathbf{Comod}_\Gamma$ if the tensor product preserves surjections.

Remark 2.5. By looking at [Lur17, p. 4.8.1], if $\mathbf{C}$ is symmetric monoidal, then Day convolution will endow $\mathrm{PSh}(\mathbf{C})$ with the unique symmetric monoidal structure such that the Yoneda embedding is symmetric monoidal and the tensor product preserves colimits in each variable.

Remark 2.6. Let us set some terminology with respect to symmetric monoidal category. Let $f : \mathbf{C} \to \mathbf{D}$ be a map of $\infty-$categories. Recall that a symmetric monoidal category is a cocartesian fibration $\mathbf{C}^{\otimes} \to \mathbf{Fin}_*$ satisfying the Segal condition. The following is an explanation of the terminology used in the coming theorems.

- When we say "**C** *can be promoted to a symmetric monoidal category*" it means that there exists a functor $\mathbf{C}^{\otimes} \to \mathbf{Fin}_*$ and an identification of the fiber over the (pointed) set $\langle 1 \rangle$ with **C**.

- Fix a symmetric monoidal category $\mathbf{C}^{\otimes} \to \mathbf{Fin}_*$ such that **C** is identified with the fiber over $\langle 1 \rangle$. Let X be the space of diagrams of the following form

$$\begin{array}{ccc} \mathbf{C}^{\otimes} & \longrightarrow & \mathbf{D}^{\otimes} \\ \downarrow & \swarrow & \\ \mathbf{Fin}_* & & \end{array}$$

  such that the vertical map preserves cocartesian edges and the fiber of $\mathbf{D}^{\otimes} \to \mathbf{Fin}_*$ can be identified with **D** and under this identification the induced map on the fibers over $\langle 1 \rangle$ is f. We say that "**D** *and f can be simultaneosly promoted to a symmetric monoidal category and symmetric monoidal functor*" or "**D** and be promoted to a symmetric monoidal category such that f is symmetric monoidal if X is non-empty. If in addition, this space is contractible we say that this promotion is unique.

The following result extends the previous remark to spherical sheaves. We say that the Day convolution is compatible with a localization $C \to S^{-1}C$ if it preserves S-local equivalences.

Theorem 2.2. [Pst18, Proposition 2.27] If **C** is an excellent $\infty-$site, then Day convolution is compatible with the localization $L : \mathrm{PSh}_{\Sigma}(\mathbf{C}) \to \mathrm{Sh}_{\Sigma}(\mathbf{C})$. Therefore, $\mathrm{Sh}_{\Sigma}(\mathbf{C})$ can be uniquely promoted to symmetric monoidal category that preserves colimits on each variable and promotes the Yoneda embedding to a symmetric monoidal functor.

2.1.3 Sheaves of spectra and its t-structure

Let **C** be a small additive $\infty-$site and denote by $\mathrm{Sh}_{\Sigma}(\mathbf{C}, \mathbf{Sp})$ its category of sheaves of spectra. We start by stating the previously announced result.

Theorem 2.3. The category $\mathrm{Sh}_{\Sigma}(\mathbf{C}, \mathbf{Sp})$ is the stabilization of $\mathrm{Sh}_{\Sigma}(\mathbf{C})$. In particular, it is a presentable stable ∞-category.

We will now define a t-structure with good properties. We start by defining the homotopy groups $\pi_n X$ of a sheaf of spectra X as the sheafification of the composite $\pi_n \circ X : \mathbf{C}^{op} \to \mathbf{Ab}$.

Definition 2.4. Let $X \in \mathrm{Sh}_{\Sigma}(\mathbf{C}, \mathbf{Sp})$, we call X connective if $\pi_n X$ for $n < 0$. In turn, we call X coconnective if $\Omega^{\infty} X$ is a discrete sheaf of spaces. We denote by $(\mathrm{Sh}_{\Sigma}(\mathbf{C}, \mathbf{Sp}))_{\geq 0}$ and $(\mathrm{Sh}_{\Sigma}(\mathbf{C}, \mathbf{Sp}))_{\leq 0}$ the full subcategories of connective and coconnective sheaves of spectra repectively.

Proposition 2.2. [Pst18, Proposition 2.16] The pair $((\mathrm{Sh}_{\Sigma}(\mathbf{C}, \mathbf{Sp}))_{\geq 0}, (\mathrm{Sh}_{\Sigma}(\mathbf{C}, \mathbf{Sp}))_{\leq 0})$ determines a right complete t-structure on $\mathrm{Sh}_{\Sigma}(\mathbf{C}, \mathbf{Sp})$ compatible with filtered colimits. Moreover, we have an equivalence $\mathrm{Sh}_{\Sigma}(\mathbf{C}, \mathbf{Sp})^{\heartsuit} \simeq \mathrm{Sh}_{\Sigma}(\mathbf{C}, \mathbf{Sp})$.

We will finish this section by discussing symmetric monoidal structures on $\mathrm{Sh}_{\Sigma}(\mathbf{C}, \mathbf{Sp})$. The following presents an important property of $\mathrm{Sh}_{\Sigma}(\mathbf{C}, \mathbf{Sp})$.

Proposition 2.3. [Pst18, Proposition 2.19] The functor $\Omega^{\infty} : \mathrm{Sh}_{\Sigma}(\mathbf{C}, \mathbf{Sp}) \to \mathrm{Sh}_{\Sigma}(\mathbf{C})$ admits a fully faithful left adjoint Σ^{∞}_{+} whose essential image is $(\mathrm{Sh}_{\Sigma}(\mathbf{C}, \mathbf{Sp}))_{\geq 0}$.

This functor will fix our induced symmetric monoidal structure oin $\mathrm{Sh}_\Sigma(\mathbf{C}, \mathbf{Sp})$. This structure will have the universal property that the data of a symmetric monoidal functor out of $\mathrm{Sh}_\Sigma(\mathbf{C}, \mathbf{Sp})$ to another symmetric monoidal stable category $\mathbf{D}$ is equivalent to the data of a symmetric monoidal functor from $\mathrm{Sh}_\Sigma(\mathbf{C})$ to $\mathbf{D}$.

Theorem 2.4. The category $\mathrm{Sh}_\Sigma(\mathbf{C}, \mathbf{Sp})$ can be promoted to a unique symmetric monoidal structure such that tensoring preserves colimits in each varible and the functor Σ^∞_+ is symmetric monoidal.

2.2 The ∞-category $\mathbf{Syn}_E$

We now return to our context of synthetic spectra for an Adams-type ring spectrum. We start by defining the latter and establish some of its properties which will be necessary to establish favourable properties of Syn_E.

2.2.1 Adams-type spectra

In this short section, we will present Adams-type spectra, which will be main characters of our study. The advantage of the restriction to this family of spectra lies in 2.5, which allows 2.5 and 2.3 in the next sections. This definition was introduced by Adams in [Ada74].

Definition 2.5. Let E be a homotopy commutative ring spectrum. We call E of Adams-type if it can be expressed as a filtered colimit $\mathrm{colim}_I E_i$ where E_i are finite E-projective spectra such that the canonical map $E^*E_i \to \mathrm{Hom}_{\pi_*E}(E_*E_i, \pi_*E)$ is an equivalence.

Remark 2.7. Since E can be expressed as a filtered colimit, one can see that $E_*E = \mathrm{colim}\, E_*E_i$ is a filtered colimit of projectives and therefore flat as a π_*E-module.

The first crucial result of this definition is the following.

Theorem 2.5 (Tor Spectral Sequence). [Ada74] Let X, Y be spectra and E a homotopy commutative ring spectrum such that E_*E is flat as π_*E-module. There exists a spectral sequence

$$\mathrm{E}^2_{s,t} \simeq \mathrm{Tor}^{s,t}_{\pi_*E}(E_*X, E_*Y) \Rightarrow E_*(X \otimes Y).$$

The second and last result used in the rest of our study is the following.

Proposition 2.4. [Pst18, Theorem 3.25] The morphism of ∞-sites $E_*(-) : \mathbf{Sp}^{\mathrm{fp}}_E \to \mathbf{Comod}^{\mathrm{fp}}_E$ admits a common envelope.

Let E be an Adams-type ring spectrum. In this section, we will take fruits of the general framework developed in the previous sections. In order to do so, one has to prove that $\mathbf{Sp}^{\mathrm{fp}}_E$ admits the structure of a excellent ∞-site.

2.2.2 The ∞-site $\mathbf{Sp}^{\mathrm{fp}}_E$

The goal of this section is to prove the following result.

Proposition 2.5. The ∞—category $\mathbf{Sp}^{\mathrm{fp}}_E$ admits a topology promoting it to an excellent ∞-site such that the homology functor $E_*(-) : \mathbf{Sp}^{\mathrm{fp}}_E \to \mathbf{Comod}^{\mathrm{fp}}_{E_*E}$ is a morphism of excellent ∞-sites.

We will start by noticing that this category is additive.

Lemma 2.1. The ∞—category $\mathbf{Sp}^{\mathrm{fp}}_E$ is additive and the homology functor $E_*(-)$ is an additive functor.

Proof. We start by proving the last claim, i.e. the homology functor preserves finite coproducts. But this follows directly from the fact that E represents a generalized homology theory which satisfies the additivity axiom. Therefore finite coproducts of finite projective spectra is finite projective. Since finite coproducts and products agree in **Sp**, one sees that the inclusion of $\mathbf{Sp}_E^{\mathrm{fp}}$ also preserves finite products. One another hand, fully faithfullness descends to homotopy categories. Thus, since $\mathrm{h}\mathbf{Sp}_E^{\mathrm{fp}}$ contains the zero object and finite products and coproducts, it follows that it is additive (as it is a full subcategory of an additive category). □

We notice that the last lemma still holds for arbitrary ring spectra, however the next lemma exemplifies the necessity of Adams-type condition.

Lemma 2.2. If $X \in \mathbf{Sp}_E^{\mathrm{fp}}$ and Y is an arbitrary spectrum, then $E_*(X \otimes Y) \simeq E_*X \otimes E_*Y$. Therefore, the smash product of spectra endows $\mathbf{Sp}_E^{\mathrm{fp}}$ with a structure of a symmetric monoidal ∞−category such that the homology functor is symmetric monoidal.

Proof. We procceed again by proving the last claim. In [Ada74, Section III.13], Adams proves that if E is Adams-type then, we have the following Kunneth spectral sequence for arbitrary spectra X and Y,

$$\mathrm{E}^2_{s,t} \simeq \mathrm{Tor}^{s,t}_{\pi_* E}(E_*X, E_*Y) \Rightarrow E_{s+t}(X \otimes Y)$$

Therefore if X is E-projective, this sequence collapses on the second page and the result follows. By this result, we have that smash product preserves finiteness and E-projectiveness. One can check that π_*E is a projective E_*E-comodule and thus the unit of the smash product is in $\mathbf{Sp}_E^{\mathrm{fp}}$.Therefore promotes $\mathbf{Sp}_E^{\mathrm{fp}}$ into a symmetric monoidal subcategory. □

Remark 2.8. [Pst18, Lemma 3.18] A minor part in the definition of excellent ∞-site is the existence of duals. One can verify that the Spanier-Whitehead dual of a finite projective spectrum is finite projective.

We finish by defining the topology on $\mathbf{Sp}_E^{\mathrm{fp}}$ in the following way: $\{Q_i \to P\}_{i \in I}$ is a covering family if it consists of a single map which is a covering after taking homology.

Proof of 2.5. We start by proving that this definition endows $\mathbf{Sp}_E^{\mathrm{fp}}$ with the structure of an ∞-site. Clearly, equivalences induce surjections on homology so they are coverings. One another another hand, compositions of surjections are surjections. One can also see that since pullbacks of surjections are surjections, therefore it suffices to prove that homology preserves pullbacks of homology surjections in $\mathbf{Sp}_E^{\mathrm{fp}}$. Let $X, Y, Z \in \mathbf{Sp}_E^{\mathrm{fp}}$, a homology surjection $X \to Y$ and a map $Z \to Y$. Notice that we have a fiber sequence $X \times_Y Z \to X \oplus Z \to Y$. By taking long exact sequence on homology and since Y is projective and $X \oplus Z \to Y$ surjective, it splits in short sequences and the result follows.

It remains to show that that smashing with any $X \in \mathbf{Sp}_E^{\mathrm{fp}}$ preserves homology surjections. But this follows from left exactness of the tensor product and the fact that homology is a symmetric monoidal functor. □

2.2.3 Main properties of Syn_E

Returning to our context of synthetic spectra, we now are able to deduce various categorical properties of Syn_E .

Proposition 2.6. The category Syn_E is a presentable stable ∞-category. Moreover, it can also be promoted to a symmetric monoidal structure in a prefered way such that its tensor product preserves colimits in every variable.

Proof. By 2.5, we can see that Syn_E is the category of spherical sheaves of spectra in an excellent site. Therefore, by 2.3 it presentable and stable, and by 2.4 it has a canonical symmetric monoidal structure such that its tensor product preserves colimits in every variable. □

Proposition 2.7. The synthetic analogue functor $\nu : \mathbf{Sp} \to \mathrm{Syn}_E$ is lax symmetric monoidal and preserves filtered colimits. Moreover, this functor is symmetric monoidal when restricted to finite E-projective spectra.

Proof. Recall that νX is defined to be $\Sigma^\infty_+ y(X)$. Since by 2.2 and 2.4, sheafification and Σ^∞_+ are symmetric monoidal and commute with colimits, it suffices to prove that the Yoneda embedding is lax symmetric monoidal and preserves filtered colimits. On the one hand, finite spectra are compact and therefore the latter claim follows. Again by 2.2, we see that Yoneda embedding restricted to $\mathbf{Sp}_E^{\mathrm{fp}}$ is symmetric monoidal. We claim that the Yoneda embedding has a symmetric monoidal left adjoint and therefore it is lax symmetric monoidal. By taking the left Kan extension of the inclusion $\mathbf{Sp}_E^{\mathrm{fp}} \subset \mathbf{Sp}$ along the Yoneda embedding for $\mathbf{Sp}_E^{\mathrm{fp}}$, one can check that this provides a left adjoint for the Yoneda embedding. Since the inclusion and the restricted Yoneda are symmetric monoidal, the left Kan extension is symmetric monoidal. □

In general, this functor doesn't preserve cofiber sequences. However, the following result is a sufficient condition for when it does.

Proposition 2.8. [Pst18, Lemma 4.23] If $A \to B \to C$ is a cofiber sequence in $\mathbf{Sp}$, then the following are equivalent:

1. $\nu A \to \nu B \to \nu C$ is a cofiber sequence,
2. $0 \to E_*A \to E_*B \to E_*C \to 0$ is a short exact sequence in $\mathbf{Comod}_{E_*E}$.

2.2.4 The t-structure on Syn_E and its heart

The goal of this subsection is to prove the following result.

Proposition 2.9. The ∞-category Syn_E admits a right complete t-tructure compatible with filtered colimits. Moreover, there exists an equivalence $\mathrm{Syn}_E^\heartsuit \simeq \mathbf{Comod}_{E_*E}$.

The first statement of this result will be purely formal in the theory of spherical sheaves following from 2.2. Moreover, by the same result, there exists an equivalence $\mathrm{Syn}_E^\heartsuit \simeq \mathrm{Sh}_\Sigma{}^{\mathbf{Set}}(\mathbf{Sp}_E^{\mathrm{fp}})$. Therefore, the second statement will follow from the following lemma.

Lemma 2.3. The morphism of ∞-sites $E_*(-) : \mathbf{Sp}_E^{\mathrm{fp}} \to \mathbf{Comod}_E^{\mathrm{fp}}$ induces an equivalence

$$E_* : \mathrm{Sh}_\Sigma(\mathbf{Sp}_E^{\mathrm{fp}}, \mathbf{Set}) \simeq \mathbf{Comod}_{E_*E}$$

This lemma will follow from a special case of a more general result that answers the following question: *When does a morphism of excellent ∞—sites induces an equivalence on spherical sheaf categories?*

Let $f : \mathbf{C} \to \mathbf{D}$ be a morphism of excellent ∞—sites, we know that we can define an induced functor $f^* : \mathrm{Sh}_\Sigma(\mathbf{C}, \mathbf{Set}) \to \mathrm{Sh}_\Sigma(\mathbf{D}, \mathbf{Set})$ induced by precomposition on presheaf categories. However, we can define the functor $f_! : \mathrm{Sh}_\Sigma(\mathbf{D}, \mathbf{Set}) \to \mathrm{Sh}_\Sigma(\mathbf{C}, \mathbf{Set})$ by taking the left Kan extension of the Yoneda embedding along the precomposition of f with the Yoneda embedding of $\mathbf{D}$. By abstract reasons, one can check that $f_!$ is left adjoint to f^*.

Theorem 2.6. [Pst18, Theorem 2.26 and Remark 2.27] Let $f : \mathbf{C} \to \mathbf{D}$ be a morphism of excellent ∞-sites. If f reflects covers and admits a common envelope, then the adjunction $f_! \dashv f^*$ is an adjoint equivalence.

Proof of 2.3. By 2.6, it suffices to show that the homology functor reflects covers and admits a common envelope. By definition, the former is satisfied. The second claim is the content of [Pst18, Theorem 3.25]. □

3 Conclusion and Preview for next talk

We will conclude by introducing one of the defining features of $\mathbf{Syn}_E$. This property stems from the fact the the functor $\nu : \mathbf{Sp} \to \mathrm{Syn}_E$ does not preserve suspensions. To control this fact, we will define bigraded spheres in the following way.

Definition 3.1. Define the (t,w)–sphere as $\mathbb{S}^{t,w} := \Sigma^{t-w}\nu\Sigma^w\mathbb{S}$. Moreover, for $X,Y \in \mathrm{Syn}_E$ we define the Y-homology and cohomology of X as $Y_{t,w}X = \pi_0 \operatorname{map}(\mathbb{S}^{t,w}, Y \otimes X)$ and $Y_{t,w}X = \pi_0 \operatorname{map}(\mathbb{S}^{t,w} \otimes Y, X)$.

For the case where $Y = \mathbb{S}^{0,0}$, we call the Y–homology of X its homotopy groups. Notice that we can issue the first remark of this section by considering the canonical colimit map $\mathbb{S}^{-1,-1} = \nu\Omega\mathbb{S} \to \Omega\nu\mathbb{S} = \mathbb{S}^{-1,-1}$. We will call this map $\tau \in \pi_{-1,-1}\mathbb{S}^{-1,0} \simeq \pi_{0,-1}\mathbb{S}^{0,0}$. The result that initiates the study of this map is the following.

Proposition 3.1. The cofiber sequence $\Sigma^{-1,0}\nu X \to \nu X \to \operatorname{cof}\tau \otimes \nu X$ identifies $\operatorname{cof}\tau \otimes \nu X \simeq (\nu X)_{\leq 0}$. Moreover since νX is connective, we have $\operatorname{cof}\tau \otimes X \in \mathrm{Syn}_E^{\heartsuit} \simeq \mathbf{Comod}_{E_*E}$.

Intuitively, this means that that the obstruction for νX taking suspensions to loops as a sheaf of spectra can be encoded as a comodule of E_*E. This map induces a filtered spectrum

$$\ldots \to \Sigma^{0,-1}\nu X \longrightarrow \nu X \longrightarrow \Sigma^{0,1}\nu X \to \ldots .$$

We call its associated spectral sequence the τ–Bockstein spectral sequence. On the other hand, we can define the spectral Yoneda embbeding $Y : \mathbf{Sp} \to \mathrm{Syn}_E$ and get the following result.

Proposition 3.2. Let X be a spectrum. In this setting, the tower by powers of τ

$$\ldots \to \Sigma^{0,-1}\nu X \longrightarrow \nu X \longrightarrow \Sigma^{0,1}\nu X \to \ldots$$

can be identified with the Whitehead tower of $Y(X)$ in Syn_E.

We remark that this implies that $Y(X)$ is therefore the localization of νX with respect to τ. Intuitively, the above proposition implies that the τ–Bockstein spectral sequence of X can be identified with the Whitehead spectral sequence of $Y(X)$. Analysing the associated graded of the τ-Bockstein filtration, we see the following result.

Proposition 3.3. [Pst18, Lemma 4.56] If X is a spectrum, then there exists an equivalence

$$\pi_{t,s}(\operatorname{cof}\tau \otimes \nu X) \xrightarrow{\simeq} \mathrm{Ext}^{s-t,s}(\pi_*E, E_*X).$$

The next talk will explain how to identify the Adams filtration of X with the above filtrations and also analyse the algebraic part of synthetic spectra shining a new light on $\mathbf{Comod}_{E_*E}$.

A Short remarks on change of rings

In this appendix, we will look at the functorial properties of the construction of synthetic spectra. Let **Adams** be the full subcategory of homotopy commutative ring spectra consisting of Adams-type spectra. We start with the following lemma.

Lemma A.1. If $f : E \to E'$ is a map of Adams type ring spectra, then every finite E_*–projective spectrum X is finite E'_*-projective. More precisely, we that the canonical map

$$E_*X \otimes_{\pi_*E} \pi_*E' \to E'_*X$$

is an isomorphism in $\mathbf{Comod}_{E'_*E'}$.

Proof. Start by noticing that $E' \otimes X \simeq E \otimes_E E' \otimes X$. Therefore we have the Kunneth spectral sequence

$$\mathrm{E}^2_{s,t} \simeq \mathrm{Tor}^{\pi_* E'}_{s,t}(\pi_*(E \otimes X), \pi_* E') \Rightarrow \pi_{t-s}(E' \otimes X)$$

which collapses. Therefore, the edge morphism is an isomorphism and since projectiveness is preserved under change of base, the lemma follows. □

Moreover, we can check that the inclusion $\mathbf{Sp}^{\mathrm{fp}}_E \subset \mathbf{Sp}^{\mathrm{fp}}_{E'}$ is a morphism of excellent ∞−sites, and therefore by [Pst18, Theorem 2.21], we have the following adjunction

$$f_! \dashv f^* : \mathrm{Syn}_E \rightleftarrows \mathrm{Syn}_{E'}$$

such that f^* is t-exact, $f_!$ is right exact and $f_!$ is the unique cocontinuous functor making the following diagram commute.

$$\begin{array}{ccc} & & \mathrm{Syn}_E \\ & \overset{Y}{\nearrow} & \downarrow f_! \\ \mathbf{Sp} & \xrightarrow[Y]{} & \mathrm{Syn}_{E'} \end{array}$$

The above remarks imply that the synthetic spectra construction is functorial $\mathrm{Syn}_{(-)} : \mathbf{Adams} \to \mathbf{Cat}_\infty$. Moreover the latter remark implies that the essential image of this functor is the nerve of a poset. This indicated that the construction depends on much less data than the ring spectrum E. Since $\mathbb{S}$ is initial in **Adams** and by the above identification of $\mathrm{Syn}_{\mathbb{S}}$ and $\mathrm{Syn}_{\mathrm{MU}}$, we recover the result that no map of spectra $\mathrm{MU} \to \mathbb{S}$ can be promoted to a map of ring spectra.

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